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Some applications of Pick's Theorem

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ABSTRACT. The article examines some properties of geometric figures represented on lattices. In particular, some applications of Pick's Theorem about the area of the simple polygons are presented.

1. INTRODUCTION

Pick's formula ties together quantities of a completely different nature. The area of an object, such as a square or a right triangle, is proportional to the product of the lengths of two of its sides. Instead, Pick's formula provides a way to measure area that does not use any multiplication. The mathematician Georg Alexander Pick published in 1899 an article in which he proved the formula that bears his name. The theorem was popularized by Hugo Steinhaus.

In teaching, there is a growing interest in geometry problems on grids. This particular sensitivity suggests the opportunity to reexamine models related to the Pick plane.

Definition 1.1. *Let us consider two families of parallel lines on the plane, dividing the plane into equal parallelograms. The set L of all intersection points of these lines (or the set of vertices of all parallelograms) is called a lattice, and the points themselves are called lattice nodes. Any of these parallelograms is called a fundamental parallelogram or a lattice-generating parallelogram.*

It is important to keep in mind that the lattice consists of points (nodes), and the straight lines themselves do not belong to it.

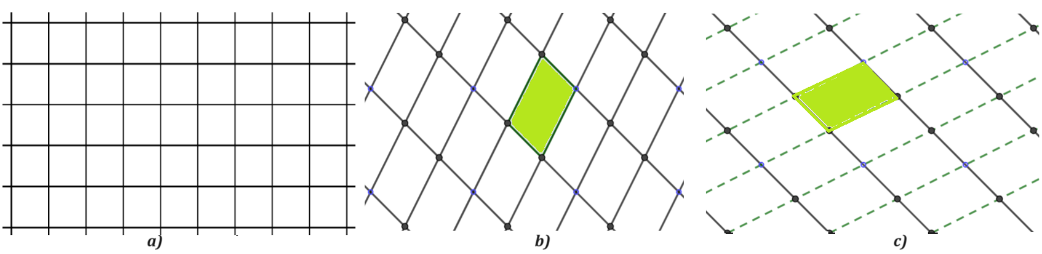

FIGURE 1. Lattices

The orthogonal integer lattice $\mathbb{Z}$, consisting of points in the Cartesian coordinate system with integer coordinates, is given by the equations $x = m$ and $y = n$, where $m, n \in \mathbb{Z}$ (Figure 1). The lattice of points is not directly related to the family of lines, unlike its fundamental parallelogram. Thus, the same family of points can be obtained by intersecting other families of lines that are not orthogonal (Figure 1 b, c).

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A lattice on a plane is a powerful tool that allows one to translate analytical problems into geometric language and back.

So, for example, we have $\mathbb{Z}^2 = L(i, j)$, where i and j are two mutually perpendicular vectors of unit length. Moreover, any lattice can be defined exactly this way by choosing two vectors that begin at the same lattice node and have their endpoints at different vertices of some fundamental parallelogram.

Definition 1.2. *Given a lattice, the triangle with vertices at the lattice nodes is called primitive if, apart from its vertices, it does not have other lattice nodes inside itself or on its sides.*

Every fundamental parallelogram can be cut diagonally into two primitive triangles. Also, two primitive congruent triangles will complement each other to a parallelogram.

The set of primitive triangles of the lattice $\mathbb{Z}^2$ does not contain acute triangles.

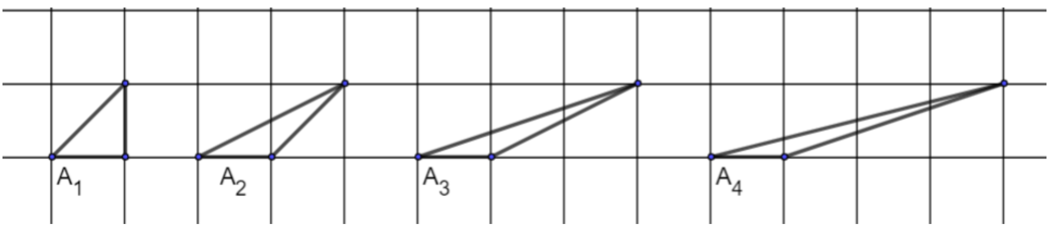

FIGURE 2. Primitive triangles

Let us start with two situations.

Example 1.1. Four crickets are placed on the vertices of a square on the orthogonal integer lattice $\mathbb{Z}^2$. Every minute, one of the crickets jumps over another cricket. The new position of the jumping cricket is symmetrical to its previous position, with the cricket it jumped over being the center of symmetry. Prove that crickets cannot simultaneously end up in the vertices of a larger square [11].

Solution. It is known that in the Cartesian coordinate system the point $M(x, y)$ symmetrical to the point with coordinates (a, b) is the point $M(2a - x, 2b - y)$. If initially the crickets are positioned at the points $(0; 0)$, $(0; 1)$; $(1; 0)$; $(1; 1)$, in the result of any symmetric jump, each cricket will end up at a point with integer coordinates.

We assume that the answer is positive, that is, at some point the crickets will be positioned in the vertices of a square with side greater than 1. Jumping in opposite order, they should get to the vertices of the smaller one. But, starting to jump from the vertices of the larger square, they will always get to the nodes of the grid consisting of large squares. In other words, the distance between them cannot be less than the side of the large square. A contradiction.

Example 1.2. Three crickets (represented as dots) are placed on three vertices of a square with a side length of 1. Each cricket can jump over one of the other two crickets to a new position that is symmetrical to its previous position, with the cricket it jumped over being the center of symmetry. It is obvious that crickets will always land on the nodes of the grid. What positions can the crickets occupy after several jumps? [11]

Solution. Consider the triangle ABC with the vertices situated in the vertices of a square grid. We will consecutively perform transformations as follows.

Step 1. We will identify symmetrical points with one of the vertices of the triangle, considering one of the other two vertices as the center of symmetry. For example, $S_C(B) = B'$. We form a new triangle $AB'C$.

Step 2. For this triangle, choosing as the center of symmetry, for example, vertex C, we repeat the operation, identifying the symmetric of vertex A: $S_C(A) = A'$. We form a new triangle $A'B'C$.

We can perform these transformations successively any number of times (Figure 3).

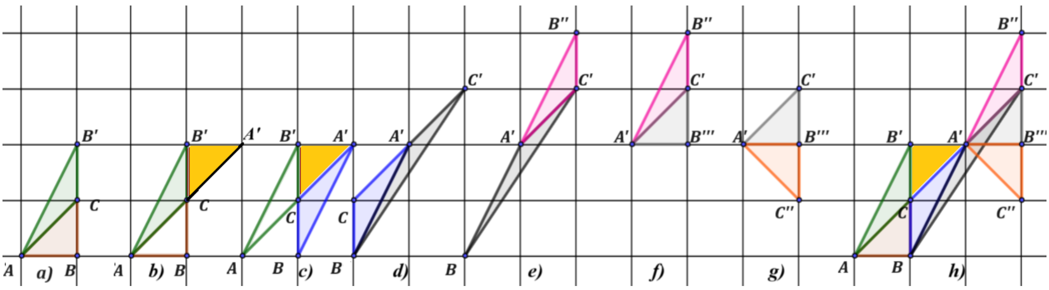

FIGURE 3. Three crickets

As a result of these transformations, the following can be observed:

1) The vertices of the triangles obtained by the named transformations will be lattice points.
2) The area of the obtained triangles is invariant.
3) The area of each triangle obtained as a result of a transformation is equal to $\frac{1}{2}$.
4) If a primitive triangle is completed to a parallelogram, then this parallelogram will contain no nodes either inside or on the sides.

Another convenient way to define a lattice on a plane is as follows.

Definition 1.3. *Let a and b be non-zero and non-collinear vectors and O the origin of the coordinate system. Then the set $L(a, b)$ of all points P such that $OP = ma + nb$, where m, n are integers, is a lattice.*

Starting with three non-zero and non-coplanar vectors, it is easier way to define lattices of points in space. Thus, the orthogonal integer lattice $\mathbb{Z}^3$ in space is obtained, for example, by choosing (in the Cartesian coordinate system) three vectors $i = (1, 0, 0)$, $j = (0, 1, 0)$, $k = (0, 0, 1)$.

Just as on a plane, a spatial lattice can be constructed starting from an arbitrary parallelepiped. Lattices can be defined similarly in spaces of higher dimensions.

Definition 1.4. *We say that a polygon is simple if all its sides connect points of the $\mathbb{Z}^2$ grid.*

The main properties of fundamental parallelograms and primitive triangles are contained in the following statements.

Theorem 1.1. *All primitive triangles on a lattice have equal areas.*

Theorem 1.2. *For the smallest distance **d** between the lattice points with the area of the fundamental parallelogram **A** the following inequality holds:*

$$\boldsymbol{d} \leq \sqrt{\frac{2\boldsymbol{A}}{\sqrt{3}}}. \tag{1.1}$$

Equality is achieved on a lattice, where the fundamental parallelogram is a rhombus with an acute angle of 60 degrees.

Definition 1.5. *Simple triangles with sides $1, 1, \sqrt{2}$ on $\mathbb{Z}^2$ are called minimal.*

Let's highlight some of the simplest properties of lattices.

1) A straight line passing through two lattice nodes contains an infinite number of lattice nodes. In this case, all neighboring nodes located on this line are equidistant.
2) The parallel translation of the plane (space), which transfers one lattice node to another node, translates the lattice onto itself.
3) The lattice is symmetric relative to the midpoint of any segment connecting two nodes of this lattice. Moreover, the midpoints of all segments ending at the nodes of a given network form a new network that includes the old one.
4) (Parallelogram rule.) If three vertices of a parallelogram are lattice nodes, then its fourth vertex is also a lattice node. In space: if four vertices of a parallelepiped that do not lie in the same plane are lattice nodes, then its remaining vertices are also lattice nodes.
5) If a parallelogram with vertices at the lattice nodes contains no other nodes on its edges or within its interior, then this is the fundamental parallelogram of the lattice. Moreover, this property is the criterion for a parallelogram to be considered fundamental.

A similar property holds for a fundamental parallelepiped in space.

The next theorem relates to the impossibility of locating a regular triangle on an integer lattice $\mathbb{Z}^2$. It was apparently proved by E. Lucas in 1878 (see [5]). His proof can be based on elementary information from the theory of divisibility of numbers or using elements of trigonometry.

Theorem 1.3. *A regular triangle cannot be placed on an integer lattice $\mathbb{Z}^2$.*

Proof. I. Let us assume that a regular triangle can be positioned on the lattice in the desired way and that one of its vertices is at the origin of the coordinate system, and its other two vertices have coordinates (a, b) and (c, d). We can assume that four integers a, b, c, d have no common divisors other than ± 1. The latter follows from the fact that the points $(0, 0)$, $(a/k, b/k)$, $(c/k, d/k)$ are also the vertices of an equilateral triangle if k is the common divisor of all four numbers (Figure 4 a,b).

$$a^2 + b^2 = c^2 + d^2 = (a - c)^2 + (b - d)^2 \tag{1.2}$$

$$a^2 + b^2 = c^2 + d^2 = 2(ac + bd) \tag{1.3}$$

$$a^2 + b^2 + c^2 + d^2 = 4(ac + bd). \tag{1.4}$$

The sum of the squares of four numbers is divisible by 4. Then either all four numbers are even or all odd. The first is impossible because these numbers, according to our choice, are mutually exclusive. The second is impossible because then the identity $a^2 + b^2 = (a - c)^2 + (b - d)^2$ is not fulfilled, because its left part is not divisible by 4, and the right part is divisible. This contradiction proves the formulated statement.

II. The second proof is based on trigonometry. Note that if two rays with origins at the origin of coordinates pass through nodes (a, b) and (c, d) of the $\mathbb{Z}^2$ lattice (Figure 4), then the tangent of the angle j between these rays is a rational number or is not defined, because

$$tan(\alpha - \beta) = \frac{tan\alpha - tan\beta}{1 + tan\alpha tan\beta} = \frac{\frac{d}{c} - \frac{b}{a}}{1 + \frac{bd}{(ac)}} = \frac{ad - bc}{ac + bd}. \tag{1.5}$$

Therefore, if we assume that there is an equilateral triangle with vertices at the nodes of the $\mathbb{Z}^2$ lattice, then two rays with origins at one of its vertices, containing the sides of the triangle, form an angle of 60^o. But tg $60^o = \sqrt{3}$, which is an irrational number, and

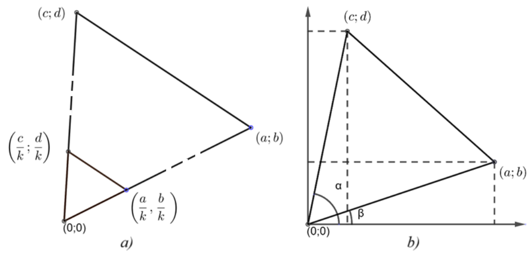

FIGURE 4. Regular triangle on the lattice

therefore it is impossible to place an equilateral triangle on the $\mathbb{Z}^2$ lattice. It is clear that a regular hexagon also cannot be located on the $\mathbb{Z}^2$ lattice, since otherwise, by connecting its vertices through one, we would get a regular triangle located on the lattice, which, as we already know, is impossible. However, one can arrange both a regular triangle and a regular hexagon in space on the $\mathbb{Z}^3$ lattice. It is enough to present a regular hexagon. The midpoints of the edges of the cube lie in the same plane and are the vertices of a regular hexagon. □

2. PICK'S THEOREM AND ITS APPLICATIONS

Theorem 2.4. *(Pick's theorem.) Consider a simple polygon P. Let i be the number of points in $\mathbb{Z}^2$ interior to the polygon and b - the number of integer points on its boundary (including both vertices and points along the sides). Then, the area A_P of the polygon can be calculated as follows: $A_P = i + \frac{b}{2} - 1$.*

The papers [1, 4, 5, 10, 11] explore various didactic aspects of the applications of Pick's theorem. The formula can be generalized to formulas for certain types of non-simple polygons.

But this formula cannot be generalized to three-dimensional space.

Pick's formula allows a simple proof for Theorem 1.3. We assume that the regular triangle is placed on the lattice $\mathbb{Z}^2$. Then, according to Pick's formula, the area of the triangle is a rational number. On the other hand, the area of the regular triangle (Fig. 4b), where a, b are integers, is an irrational number

$$\mathbf{A} = \frac{a^2 + b^2}{4}\sqrt{3}. \tag{2.6}$$

It follows that the assumption was wrong, i.e. the regular triangle cannot be placed on the lattice $\mathbb{Z}^2$.

Example 2.3. The midpoints of the sides of a square are connected by segments to the vertices as shown in Figure 5. Find the ratio of the area of the square to the area of the octagon formed by these segments [11].

Proof. Since we need to find the ratio of the areas of the two surfaces, the dimensions of the squares are irrelevant. So, let's consider a square 12×12 located on a $\mathbb{Z}^2$ grid; the edges of the square are on straight grids. We find that all vertices of the octagon are connected to the nodes of the grid; moreover, from here it is easy to see that this octagon is not regular (Figure 5) — it is equilateral, but not equiangular. From Pick's formula it now easily follows that the area of the octagon is $21 + 8/2 - 1 = 24$. Therefore, the requested ratio of areas is 6. □

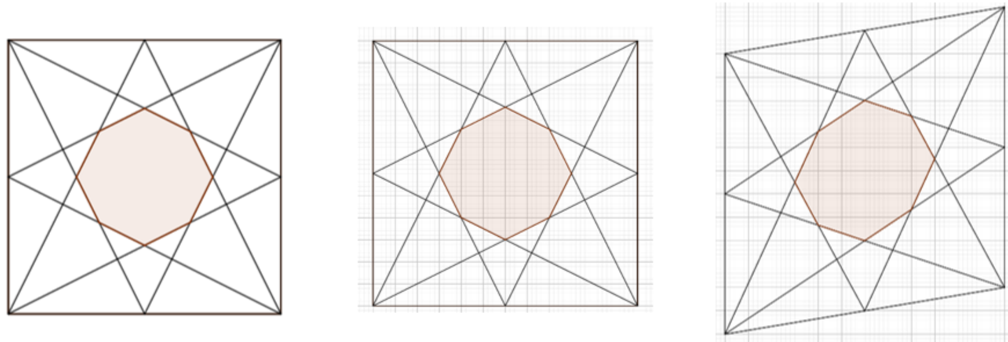

FIGURE 5. Octagon in a square

A basic result from vector geometry, which we will not prove here, states that given any three non-collinear points A, B, C in the plane, every vector $\overline{OQ}$ may be represented in the form

$$\overline{OQ} = r\overline{OA} + s\overline{OB} + t\overline{OC} \tag{2.7}$$

for unique r, s, t, satisfying

$$r + s + t = 1. \tag{2.8}$$

Moreover, Q is in the interior of the ΔABC if and only if such r, s, t are all positive. In similar fashion, any point on the line through A and B can be expressed as $r\overline{OA} + s\overline{OB}$ with $r+s=1$, and lies between A and B if and only if $r,\ s > 0$.

Example 2.4. *Let a lattice point A in $\mathbb{Z}^2$ be given and OA contains no other lattice points.The point P is the point on the lattice closest to the line OA. For every lattice point Q the vector $\overline{OQ}$ can be expressed in the form $n\overline{OP} + k\overline{OA}$ for some pair of integers (n, k). Moreover, when $\overline{OQ}$ is expressed in such form, we have $n = 2A_{\Delta OQA}$.*

Proof. If P is the point closest to line OA, it follows that there are no lattice points inside the ΔOPA or on its boundary, other than O, P, A themselves, since any such point would be closer than P to line OA. Therefore, by Pick's Theorem, $A_{\Delta OPA} = \frac{1}{2}$.

Let Q be a lattice point. By Pick's Theorem, $A_{\Delta OQA} = \frac{n}{2}$ for some integer n. Thus $A_{\Delta OQA} = n \cdot A_{\Delta OPA}$. By the base–height formula for the area of a triangle, it follows that Q is on the line parallel to line OA that passes through the endpoint of the vector $n\overline{OP}$. Thus $\overline{OQ} = n\overline{OP} + k\overline{OA}$ for some real k, where the endpoint of $k\overline{OA}$ is a lattice point. We assert that k is an integer. Indeed, if $\{k\}$ denotes the fractional part of k, then $\{\text{k}\} = k - [k]$ and the endpoint of the vector $\{\text{k}\}\overline{OA}$ is a lattice point which lies on segment OA. Since OA has no lattice points, we must have $k = 0$. This completes the proof of the lemma. □

Example 2.5. *(IMO 1987, P5) Let $n \geq 3$ be an integer. Prove that there is a set of n points in the plane such that the distance between any two points is irrational and each set of three points determines a non-degenerate triangle with rational area.*

Proof. By Pick's theorem we can choose any n lattice points and their area immediately is rational (in fact even half-integer so in fact by scaling the coordinates of the points by a factor of 2 we can get the triangles to have natural area). So we now only have to properly select n lattice points such that the distance between any 2 of them is irrational. This can be done in various ways. For example:

1.) Take n points on the parabola $y = x^2$.

The distances between them are not rational because using that the distance between $(x_1; y_1)$ and $(x_2; y_2)$ is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$, we get:

$$d[(a, a^2), (b, b^2)] = (a - b)^2 + (a^2 - b^2)^2 = (a - b)^2(1 + (a + b)^2) \tag{2.9}$$

that is not a perfect square. Even though this construction is straightforward to present and the perfect square factors out cleanly, it's not easy to come up with. The next example is slightly more challenging to write rigorously but is much easier to conceptualize.

2.) Let C be an integer much larger than n. Then define the points $A_1 = (C + 1, C^{C+1})$, $A_2 = (C + 2, C^{C+2})$, ... , $A_n = (C + n, C^{C+n})$. Of course the triangles will not be degenerate. If C is large enough, $d(A_i, A_j)^2 = (i - j)^2 + (C^{C+i} - C^{C+j})^2$ is an integer but not a perfect square because as we chose C to be large the second square is significantly bigger than the first. So we can see that:$(C^{C+i} - C^{C+j})^2 < d(A_i, A_j)^2 = (i-j)^2+(C^{C+i}-C^{C+j})^2 < (C^{C+i}-C^{C+j}+1)^2$. In fact, even though these constructions all satisfy the condition, it is unlikely to guess them on the first try. A more direct and approachable method is to construct a set by induction.

□

Example 2.6. *(24th Bay Area Mathematical Olympiad, Mar 1, 2023, https://www.bamo.org/)* *A lattice point in the plane is a point with integer coordinates. Let T be a triangle in the plane whose vertices are lattice points, but with no other lattice points on its sides. Furthermore, suppose T contains exactly four lattice points in its interior. Prove that these four points lie on a straight line.*

Solution 1: We will use Pick's Theorem. The triangle T described in the problem must have area $4 + \frac{3}{2} - 1 = \frac{9}{2}$. With no loss of generality, let us assume T has one vertex at the origin O, which we identify with the zero vector. Call the other two vertices A and B. Of the four lattice points in the interior of T, let P be the point closest to line OA. It follows that there are no lattice points lying inside the ΔOPA or on its boundary, other than O, P, A themselves, since any such point would be closer than P to line OA. Therefore, by Pick's Theorem, the ΔOPA has area $\frac{1}{2}$.

As already noted, $A_{\Delta OAB} = \frac{9}{2}$. Thus, by the Example 2.4, $\overline{OB} = 9\overline{OP} - k\overline{OA}$ for some integer k (the minus sign in the expression is not a typo, but a deliberate convenience for what follows). Rearranging, and using the fact that $\overline{OO}$ is the zero vector, we have $\overline{OP} = \frac{k}{9}\overline{OA} + \frac{1}{9}\overline{OB} + \frac{8-k}{9}\overline{OO}$. Since P is in the interior of ΔOAB, we have $0 < k < 8$. We will consider the possible values of k in turn.

If $k \equiv 0(mod 3)$, then $\frac{1}{3}\overline{OB} = 3\overline{OP} - \frac{k}{3}\overline{OA}$ is a lattice point lying on segment OB. This contradicts the specification of T as having no lattice points on its sides.

If $k \equiv 2(mod 3)$, then $\frac{1}{3}\overline{OB} + \frac{2}{3}\overline{OA} = 3\overline{OP} - \frac{k-2}{3}\overline{OA}$ is a lattice point lying on AB, similarly yielding a contradiction. The remaining possibilities are $k = 1, 4, 7$.

If $k = 1$, then the interior of T contains in its interior the four collinear lattice points $\overline{OP} = \frac{1}{9}\overline{OA} + \frac{1}{9}\overline{OB} + \frac{7}{9}\overline{OO}$, $2\overline{OP} = \frac{2}{9}\overline{OA} + \frac{2}{9}\overline{OB} + \frac{5}{9}\overline{OO}$, $3\overline{OP} = \frac{3}{9}\overline{OA} + \frac{3}{9}\overline{OB} + \frac{3}{9}\overline{OO}$, $4\overline{OP} = \frac{4}{9}\overline{OA} + \frac{4}{9}\overline{OB} + \frac{1}{9}\overline{OO}$.

If $k = 4$, then the interior of T contains in its interior the four collinear lattice points $\overline{OP} = \frac{4}{9}\overline{OA} + \frac{1}{9}\overline{OB} + \frac{4}{9}\overline{OO}$, $3\overline{OP} - \overline{OA} = \frac{3}{9}\overline{OA} + \frac{3}{9}\overline{OB} + \frac{3}{9}\overline{OO}$, $5\overline{OP} - 2\overline{OA} = \frac{2}{9}\overline{OA} + \frac{5}{9}\overline{OB} + \frac{2}{9}\overline{OO}$, $7\overline{OP} - 3\overline{OA} = \frac{1}{9}\overline{OA} + \frac{7}{9}\overline{OB} + \frac{1}{9}\overline{OO}$.

If $k = 7$, then the interior of T contains in its interior the four collinear lattice points $\overline{OP} = \frac{7}{9}\overline{OA} + \frac{1}{9}\overline{OB} + \frac{1}{9}\overline{OO}$, $2\overline{OP} - \overline{OA} = \frac{5}{9}\overline{OA} + \frac{2}{9}\overline{OB} + \frac{2}{9}\overline{OO}$, $3\overline{OP} - 2\overline{OA} = \frac{3}{9}\overline{OA} +$

$\frac{3}{9}\overline{OB}+\frac{3}{9}\overline{OO}$, $4\overline{OP}-3\overline{OA}=\frac{1}{9}\overline{OA}+\frac{4}{9}\overline{OB}+\frac{4}{9}\overline{OO}$. Thus, the four lattice points inside T are collinear in every case, as desired.

Solution 2: We assume the same basic facts about vectors, as well as Pick's Theorem and the determinant formula for the area of a parallelogram.

Let T have vertices $A(x_1,y_1)$, $B(x_2,y_2)$, and $C(x_3,y_3)$.

We know that

$$2A_{\triangle ABC}=\left\|\begin{matrix}x_2-x_1 & y_2-y_1\\ x_3-x_1 & y_3-y_1\end{matrix}\right\|=9 \tag{2.10}$$

Consider equation (2.10) modulo 3, that is, over the field $\mathbb{Z}/3\mathbb{Z}$. In this setting, the determinant is zero, so the vectors $u=(x_2-x_1,y_2-y_1)$ and $v=(x_3-x_1,y_3-y_1)$ are linearly dependent. If either of these vectors is zero (mod 3, that is), or if they are equal, then the trisection points of a side of T are lattice points, which contradicts the problem statement. Thus $u,v\neq 0$ and $u=-v$. An immediate consequence is that $(x_1+x_2+x_3,y_1+y_2+y_3)=u+v+3(x_3,y_3)=0$ over $\mathbb{Z}/3\mathbb{Z}$, with the result that the centroid, $\overline{OG}=1/3(\overline{OA}+\overline{OB}+\overline{OC})$, is a lattice point. Now consider the triangle ABG, whose area is $1/3A_{\triangle ABC}=3/2$. By Pick's Theorem, the triangle ABG has either:

- one lattice point in its interior and none on its boundary (besides vertices), or
- two lattice points on its boundary.

Case 1: The triangle ABG has a lattice point in its interior and none on its boundary. In this case, a repetition of the preceding (mod 3) argument shows that the centroid G_1 of the triangle ABG is a lattice point. In this case, $\overline{OG_1}+k(\overline{OG}-\overline{OG_1})$ for $k=0,1,2,3$ are four collinear lattice points inside T.

Case 2: ABG has two lattice points on its boundary. Note that if at least two lattice points occur on a line, then the lattice points on that line occur at regular intervals. Thus the two lattice points on the boundary of ABG are either the midpoints of AG and BG or the trisection points of one of these sides (say, AG).

In the two cases, if we extend side AG beyond G, the next lattice point occurring on the extension is respectively either on T (at the midpoint of side BC), which is a contradiction, or inside T, being then the fourth collinear lattice point inside T.

Example 2.7. *Let V_1 be a convex polyhedron in a three-dimensional system of coordinates, whose vertices all have integer coordinates. Let V_k be the polyhedron whose vertices' radius vectors are obtained by multiplying the radius vectors of the vertices of V_1 by k. Let $N(V)$ be the number of points with integer coordinates located inside the polyhedron V or on its surface, and by $\mu(V)$ - volume of the polyhedron V. Prove that*

$$N(V_3)-3N(V_2)+3N(V_1)-1=6\mu(V_1). \tag{2.11}$$

Proof. We can observe that on a straight line (in the one-dimensional space) the next formula is correct

$$N(V_1)-1=l(V_1), \tag{2.12}$$

where $l(V_1)$ is the length of the segment V_1, the extremities of which have integer coordinates.

Let us show that for a convex polygon V_1 on the plane, with vertices at the points of an integer lattice the following formula holds

$$N(V_2)-2N(V_1)+1=2A_{V_1}, \tag{2.13}$$

where A_{V_1} is the area V_1.

Formula (2.12) shows that (2.13) is true for degenerate two - dimensional polygons-segments. From here it immediately follows that if the polygon V_1 is divided by a diagonal into two parts V_1' and V_1'' then

$$N(V_2) - 2N(V_1) + 1 = [N(V_2') - 2N(V_1') + 1] + [N(V_2'') - 2N(V_1'') + 1]. \tag{2.14}$$

Therefore, it all comes down to proving (2.13) for triangles. Let us place the origin O at

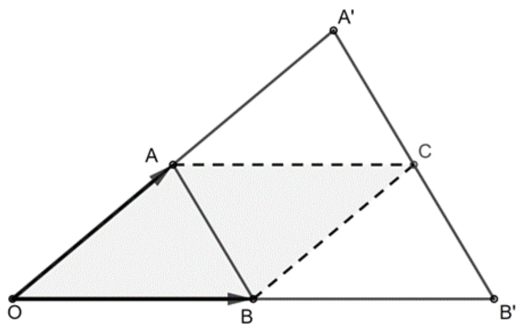

FIGURE 6. Double triangle area

the vertex of the integer triangle V_1 (Figure 6) and let A and B be its two other vertices. Let $OA'B'$ be triangle V_2. From Figure 6 it is easy to find that the left side of formula (2.13) is equal to the number $N(\Pi)$ of integer points in the figure Π, which is a parallelogram $OACB$ without sides AC and BC.

If A_Π is the area of the parallelogram $OACB$ (equal to $2A_{V_1}$, then to prove (2.13) it remains to check that $N(\Pi) = A_\Pi$ and this fact is proven by Pick's Theorem.

Let's move on to the three-dimensional case. It is enough to prove the formula (2.11) only for tetrahedron, where the proof is similar to the two-dimensional case, but more complex additional constructions will be required (Figure 7). From the polyhedron $V_3 = OA'B'C'$, three polyhedrons of type V_2 located "in the corners" A', B', C' are thrown out, their pairwise intersections (of type V_1) are added and one extra point P is thrown out. The remainder is a parallelepiped Π with volume $6\mu(V_1)$ without part of the surface, and using these parallelepipeds the entire space can be paved without intersections. Therefore $N(\Pi) = \mu(\Pi)$, which proves the required statement.

Some interesting generalizations of Pick's formula were obtained by J. Reeve [9].

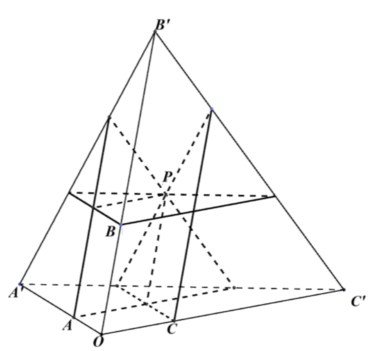

FIGURE 7. Tetrahedron

□

3. CONCLUSIONS

The simple lattice served as a starting point for K. Gauss to compare the area of a circle with the number of points with integer coordinates located inside it. What regular polygons can be placed on a lattice so that all its vertices fall on the nodes of the lattice? The G. Pick's formula, that is close connected with the well-known combinatorial formula of L. Euler on graphs, gives an answer to this question. The plane lattice is a powerful tool that allows translating analytical problems into geometric language and vice versa. In this context, interesting and instructive things become evident in the process of solving problems encountered at various mathematical competitions.

For other related developments, see also [2], [3], [8] and references cited there.

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