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Fixed point iterations for non-expansive maps and their applications to constrained minimization and feasibility problems in Hilbert spaces

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ABSTRACT. The purpose of this paper is to introduce some fixed point iterative schemes and prove that they converge faster than other iterations in the literature. This paper introduces three novel modified multistep iterative schemes (A), (B) and (C). Fixed point theorems are proven with these newly introduced multistep iterative schemes for the class of contraction mappings with fixed point $p = Tp$ and non-expansive mappings respectively. The rate of convergence was demonstrated numerically with the help of Python programs and the results showed that our modified iterative scheme (C) converged in lesser number of iterations than existing iterative schemes in the literature. With the help of well constructed theorems, these modified multistep iterative schemes were applied to constrained minimization and split feasibility problems for the class of non-expansive mappings in real Hilbert spaces.

1. INTRODUCTION AND PRELIMINARIES

Fixed point equations are related to a lot of physical problems in applied sciences and related fields that can be written as functional equations. It is always of interest to construct iterative schemes that can approximate the unique solution of the fixed point equations with lesser number of steps. Fixed point equations have vast applications in constrained minimization and split feasibility problems in Hilbert spaces. It is vital to employ faster iterative schemes to obtain solution of nonlinear functions, especially those that can be applied to real life situations like constrained minimization and split feasibility problems. Most of the papers from the present list of references presented in this work dealt with fixed point problems in Hilbert, normed and metric spaces. Some very important classes of fixed point results are the contractive-type ones. Several convergence, rate of convergence, equivalence, data dependence theorems and optimization related problems were stated and proved in framework of Hilbert and Banach spaces and a huge literature is devoted to them, for details, see ([1], [2], [3] - [25], [26], [27], [28], [29], [30], [31], [32]).

The purpose of this paper is to introduce some fixed point iterative schemes and prove that they converge faster than all of Thakur et al [31], Abbas and Nazir [1], Noor [16], Agarwal et al [2]., Ishikawa [13], Khan [14], Mann [15] and Picard [27] iterations.

We present some related definitions and Lemmas to our work as follows:
Let H be a Hilbert space, $T : H \to H$ a nonlinear mapping. The set of fixed points of T denoted by $F(T)$ is $F(T) = \{x \in H : Tx = x\}$.

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Definition 1.1. [1] *Let H be a real Hilbert space. A mapping $T : H \to H$ is said to be an averaged mapping if and only if T can be written as the average of the identity and a nonexpansive mapping, that is , $T = (1-\alpha)I + \alpha S$, where $\alpha \in [0,1]$ and $S : H \to H$ is a nonexpansive mapping.*

Definition 1.2. [1] *Let H be a real Hilbert space and C be a nonempty, closed and convex subset of H. A mapping $T : C \to C$ is said to be demi-closed at 0, if for any sequence $\{x_n\} \subset C$ which converges weakly to x and $lim_{n\to\infty}\|x_n - Tx_n\| = 0$, then $Tx = x$.*

Definition 1.3. [1] *Let H be a real Hilbert space and C be a nonempty, closed and convex subset of H. A mapping $T : C \to C$ is called:*
(i) nonexpansive if $\|Tx - Ty\| \leq \|x - y\|,\ \forall x, y \in C$.
(ii) quasi nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - x^*\| \leq \|x - x^*\|,\ \forall x \in C,\ x^* \in F(T)$.

Definition 1.4. [1] *Let H be a real Hilbert space and C be a nonempty, closed and convex subset of H. Let $T : C \to H$ be a nonlinear operator, then T is called:*
(i) a monotone operator if $< Tx - Ty, x - y > \geq 0, \forall x, y \in C$.

(ii) a $\lambda-$ strongly monotone operator if there exists $\lambda > 0$ such that

$$< Tx - Ty, x - y > \quad \geq \quad \lambda\|x - y\|^2,\ \forall x, y \in C.$$

(iii) $\beta - ism$ ($\beta-$ inverse strongly monotone) if there exists a constant $\beta > 0$ such that $< Tx - Ty, x - y > \geq \beta\|Tx - Ty\|^2,\ \forall x, y \in C$.
(iv) pseudo monotone if $< Tx, y - x > \geq 0 \Rightarrow < Ty, y - x > \geq 0,\ \forall x, y \in C$.
(v) quasi-monotone if $< Ty, x - y >> 0 \Rightarrow < Tx, x - y > \geq 0,\ \forall x, y \in C$.

In 1922, Stefan Banach employed contraction condition to obtain unique fixed point in the celebrated Banach contraction principle which is remarkable in its simplicity, but it is perhaps the most widely applied fixed point theorem in all of analysis with special applications to the theory of differential and integral equations. Let X be a complete metric space and $T : X \to X$ a self-map. T is called:
Banach contraction mapping if there exists δ, satisfying $\delta \in [0,1)$ such that

$$d(Tx, Ty) \quad \leq \quad \delta d(x, y),\ \forall x, y \in X. \tag{1.1}$$

In the framework of Banach space, we have the following definitions: Let E be a Banach space and $T : E \to E$ a self-map. T is called an $L-$Lipschitzian mapping if there exists a constant $L \geq 0$ such that

$$\|Tx - Ty\| \quad \leq \quad L\|x - y\|,\ \forall x, y \in E. \tag{1.2}$$

T is called a Banach contraction mapping if there exists $L = \delta$, satisfying $\delta \in [0,1)$ such that, (1.1) becomes

$$\|Tx - Ty\| \quad \leq \quad \delta\|x - y\|,\ \forall x, y \in E. \tag{1.3}$$

T is called a nonexpansive mapping if there exists $L = \delta = 1$, in which case, (1.3) becomes

$$\|Tx - Ty\| \quad \leq \quad \|x - y\|,\ \forall x, y \in E. \tag{1.4}$$

T is called Zamfirescu mapping if there exists $\delta \in [0,1)$ such that

$$\|Tx - Ty\| \quad \leq \quad \delta\|x - y\| + 2\delta\|x - Tx\|,\ \forall x, y \in E. \tag{1.5}$$

In 1972, Zamfirescu [32], generalize the Banach fixed point theorem using the following contractive condition (1.3).

T is called Osilike mapping if there exists $\delta \in [0,1)$ and $L \geq 0$ such that

$$\|Tx - Ty\| \quad \leq \quad \delta\|x - y\| + L\|x - Tx\|,\ \forall x, y \in E. \tag{1.6}$$

Osilike [23] proved several stability results using contractive definition (1.6).

T is called contractive-like mapping if there exists $\delta \in [0,1)$ and a monotone increasing function $\varphi : R^+ \to R^+$ with $\varphi(0) = 0$ such that

$$\|Tx - Ty\| \leq \delta\|x - y\| + \varphi(\|x - Tx\|),\ \forall x, y \in E. \tag{1.7}$$

Let F_T denote the set of all fixed point of T. That is, $F_T = \{p : p = Tp\}$.
In 2003, Imoru and Olatinwo [12] proved some stability results using the general contractive definition (1.5).

Bosede and Rhoades [9] made the following assumption to obtain a general class of mapping and proved fixed point theorem for Picard and Mann iterations. That is, if $x = p$ (is a fixed point) and $\delta \in [0,1)$ then, (1.3), (1.5), (1.6) and (1.7) becomes

$$\|p - Ty\| \leq \delta\|p - y\|,\ \forall x, y \in E. \tag{1.8}$$

Chidume and Olaleru [10] gave examples to show that the class of mappings satisfying (1.8) is more general than that of (1.3), (1.5), (1.6) and (1.7) provided the fixed point exists. The authors [10] also proved that every contraction map with a fixed point satisfies inequality (1.8) in the following example:

Example 1.1. *Let $E = l_\infty$, $B := \{x \in l_\infty : \|x\| \leq 1\}$ and let $T : E \to B \subseteq E$ be defined by*
$Tx = \frac{11}{12}(0, x_1^2, x_2^2, x_3^2, ...)$, *if* $\|x\|_\infty \leq 1$,
$Tx = \frac{11}{12\|x\|_\infty^2}(0, x_1^2, x_2^2, x_3^2, ...)$, *if* $\|x\|_\infty > 1$,*for* $x0 = (x_1, x_2, x_3, ...) \in l_\infty$.
Then $Tp = p$, if and only if $p = 0$.
We compute as follows:
$\|Tx - p\|_\infty = \frac{11}{12}\|(0, x_1^2, x_2^2, x_3^2, ...)\|_\infty$, *if* $\|x\|_\infty \leq 1$,
$\|Tx - p\|_\infty = \frac{11}{12\|x\|_\infty^2}\|(0, x_1^2, x_2^2, x_3^2, ...)\|_\infty$, *if* $\|x\|_\infty > 1$, *so that*
$\|Tx - p\|_\infty = \frac{11}{12}\|x\|_\infty^2 \leq \frac{11}{12}\|x\|_\infty$, *if* $\|x\|_\infty \leq 1$,
$\|Tx - p\|_\infty = \frac{11}{12}.1$, *if* $\|x\|_\infty > 1$. *Hence, we obtain that*
$\|Tx - p\|_\infty = \frac{11}{12}\|x - p\|_\infty$, *for every* $x \in l_\infty$, $p = 0$.
Thus, T satisfies contractive condition (1.6). But the map T is not a contraction. To see this, take $x = (\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, ...)$; $y = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, ...)$. *Then,*
$\|x - y\|_\infty = \frac{1}{4}$; $\|Tx - Ty\|_\infty = \frac{11}{12}\|(0, \frac{5}{16}, \frac{5}{16}, ...)\|_\infty = \frac{55}{192}$.
Suppose there exists $\delta \in [0,1)$ such that $\|Tx - Ty\|_\infty \leq \delta\|x - y\|_\infty$ for every $x, y \in E$, then we must have $\frac{55}{192} \leq \frac{\delta}{4}$ which yields that $\delta \geq \frac{220}{192} > 1$, a contradiction. So, T is not a contraction map.

Let E be a Banach space, C a closed convex subset of E and $T : C \to C$ a selfmap of C. Then,
the Picard [27] iterative scheme $\{x_n\}_{n=1}^\infty \subset C$ is defined by:

$$\begin{aligned} &x_1 \in C, \\ &x_{n+1} = Tx_n,\ n = 1, 2, ..., \end{aligned} \tag{1.9}$$

the Mann [15] iterative scheme $\{x_n\}_{n=1}^\infty \subset C$ is defined by:

$$\begin{aligned} &x_1 \in C, \\ &x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n,\ n = 1, 2, ..., \end{aligned} \tag{1.10}$$

where the sequence $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$.

the Picard-Mann hybrid [14] iterative scheme $\{x_n\}_{n=1}^{\infty} \subset C$ is defined by:

$$\begin{aligned} &x_1 \in C, \\ &x_{n+1} = T y_n, \\ &y_n = (1-\alpha_n)x_n + \alpha_n T x_n,\ n = 1, 2, ..., \end{aligned} \tag{1.11}$$

where the sequence $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$.

the Ishikawa [13] iterative scheme $\{x_n\}_{n=1}^{\infty} \subset C$ is defined by:

$$\begin{aligned} &x_1 \in C, \\ &x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ &y_n = (1-\beta_n^1)x_n + \beta_n^1 T x_n,\ n = 1, 2, ..., \end{aligned} \tag{1.12}$$

where the sequences $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n^1\}_{n=1}^{\infty} \subset (0,1)$.

the Agarwal et al. [2] iterative scheme $\{x_n\}_{n=1}^{\infty} \subset C$ is defined by:

$$\begin{aligned} &x_1 \in C, \\ &x_{n+1} = (1-\alpha_n)T x_n + \alpha_n T y_n, \\ &y_n = (1-\beta_n^1)x_n + \beta_n^1 T x_n,\ n = 1, 2, ..., \end{aligned} \tag{1.13}$$

where the sequences $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n^1\}_{n=1}^{\infty} \subset (0,1)$.

the Noor [16] iterative scheme $\{x_n\}_{n=1}^{\infty} \subset C$ is defined by:

$$\begin{aligned} &x_1 \in C, \\ &x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ &y_n = (1-\beta_n^1)x_n + \beta_n^1 T z_n, \\ &z_n = (1-\beta_n^2)x_n + \beta_n^2 T x_n,\ n = 1, 2, \ldots, \end{aligned} \tag{1.14}$$

where the sequences $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n^1\}_{n=1}^{\infty}, \{\beta_n^2\}_{n=1}^{\infty} \subset (0,1)$.

the Abbas and Nazir [1] iterative scheme $\{x_n\}_{n=1}^{\infty} \subset C$ is defined by:

$$
\begin{aligned}
&x_1 \in C, \\
&x_{n+1} = (1-\alpha_n)Ty_n + \alpha_n Tz_n, \\
&y_n = (1-\beta_n^1)Tx_n + \beta_n^1 Tz_n, \\
&z_n = (1-\beta_n^2)x_n + \beta_n^2 Tx_n,\ n = 1, 2, \dots,
\end{aligned} \tag{1.15}
$$

where the sequences $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n^1\}_{n=1}^{\infty}, \{\beta_n^2\}_{n=1}^{\infty} \subset (0,1)$.

the Thakur et al. [31] iterative scheme $\{x_n\}_{n=1}^{\infty} \subset C$ is defined by:

$$
\begin{aligned}
&x_1 \in C, \\
&x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n Ty_n, \\
&y_n = (1-\beta_n^1)z_n + \beta_n^1 Tz_n, \\
&z_n = (1-\beta_n^2)x_n + \beta_n^2 Tx_n,\ n = 1, 2, \dots,
\end{aligned} \tag{1.16}
$$

where the sequences $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n^1\}_{n=1}^{\infty}, \{\beta_n^2\}_{n=1}^{\infty} \subset (0,1)$.

The following Lemmas will be needed in proving the main results.

Lemma 1.1. *Let δ be a real number satisfying $0 \leq \delta < 1$ and $\{\epsilon_n\}_{n=0}^{\infty}$ a sequence of positive numbers such that $lim_{n\to\infty}\epsilon_n = 0$, then for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying $u_{n+1} \leq \delta u_n + \epsilon_n$, n=0,1,2,..., we have $lim_{n\to\infty}u_n = 0$.*

Lemma 1.2. *Suppose that E is a uniformly convex Banach space and $0 < q \leq t_n \leq p < 1$ $\forall\ n \in N$. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be two sequences of E such that $limsup_{n\to\infty}\|x_n\| \leq r$, $limsup_{n\to\infty}\|y_n\| \leq r$ and $lim_{n\to\infty}\|t_n x_n + (1-t_n)y_n\| = r$ hold for some $r \geq 0$. Then $lim_{n\to\infty}\|x_n - u_n\| = 0$.*

Lemma 1.3. *Let E be a uniformly convex Banach space and let C be a non-empty closed convex subset of E. Let T be a nonexpansive mapping of C into itself. Then $I - T$ is demiclosed with respect to zero.*

2. MAIN RESULTS 1

2.1. **Convergence Results For a General Class of Map in Banach Spaces.**

In this section, we introduce three types of modified multistep iterative schemes and prove strong convergence fixed point result for a general class of map.

Let E be a Banach space, C a closed convex subset of E and $T : C \to C$ a self-map of C. Then,
the modified multistep iterative scheme (A) $\{x_n\}_{n=1}^{\infty} \subset C$ is defined by:

$$
\begin{aligned}
&x_1 \in C, \\
&x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n Ty_n^1, \\
&y_n^i = (1-\beta_n^i)x_n + \beta_n^i Ty_n^{i+1},\ i = 1, 2, \dots, k-2, \\
&y_n^{k-1} = (1-\beta_n^{k-1})x_n + \beta_n^{k-1} Tx_n,\ k = 2, 3, \dots,\ n = 1, 2, \dots,
\end{aligned} \tag{2.17}
$$

where the sequences $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n^i\}_{n=1}^{\infty} \subset (0,1)$, for $(i = 1, 2, ..., k-1)$.

The modified multistep iterative scheme (B) $\{x_n\}_{n=1}^{\infty} \subset C$ is defined by:

$$x_1 \in C,\quad x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n Ty_n^1,\quad y_n^i = (1-\beta_n^i)y_n^{i+1} + \beta_n^i Ty_n^{i+1},\ i = 1,2,\dots,k-2,\quad y_n^{k-1} = (1-\beta_n^{k-1})x_n + \beta_n^{k-1}Tx_n,\ k = 2,3,\dots,\ n = 1,2,\dots, \tag{2.18}$$

where the sequences $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n^i\}_{n=1}^{\infty} \subset (0,1)$, for $(i = 1,2,...,k-1)$.

The modified multistep iterative scheme (C) $\{x_n\}_{n=1}^{\infty} \subset C$ is defined by:

$$x_1 \in C,\quad x_{n+1} = (1-\alpha_n)Ty_n^1 + \alpha_n Ty_n^1,\quad y_n^i = (1-\beta_n^i)y_n^{i+1} + \beta_n^i Ty_n^{i+1},\ i = 1,2,\dots,k-2,\quad y_n^{k-1} = (1-\beta_n^{k-1})x_n + \beta_n^{k-1}Tx_n,\ k = 2,3,\dots,\ n = 1,2,\dots, \tag{2.19}$$

where the sequences $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n^i\}_{n=1}^{\infty} \subset (0,1)$, for $(i = 1,2,...,k-1)$.
We will now prove our main results using iterative schemes (2.17), (2.18) and (2.19) as follows:

Theorem 2.1. *Let E be a Banach space, C a closed convex subset of E and $T : C \to C$ a selfmap of C with a fixed point p satisfying the condition*

$$\|p - Ty\| \le \delta\|p - y\|,\ \forall x, y \in C, \tag{2.20}$$

for some $\delta \in [0,1)$. For $x_1 \in C$, let $\{x_n\}_{n=1}^{\infty}$ be the modified multistep iterative scheme (A) defined by (2.17), where the sequences $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n^i\}_{n=1}^{\infty} \subset (0,1)$, for $(i = 1,2,...,k-1)$. Then $\{x_n\}_{n=1}^{\infty}$ (2.17) converges strongly to the unique fixed point p of T.

Proof. In view of (2.17) and (2.20), we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1-\alpha_n)Tx_n + \alpha_n Ty_n^1 - p\| \\ &\le (1-\alpha_n)\|Tx_n - p\| + \alpha_n\|Ty_n^1 - p\| \\ &\le \delta(1-\alpha_n)\|x_n - p\| + \delta\alpha_n\|y_n^1 - p\|. \end{aligned} \tag{2.21}$$

$$\begin{aligned} \|y_n^1 - p\| &\le (1-\beta_n^1)\|x_n - p\| + \beta_n^1\|Ty_n^2 - p\| \\ &\le (1-\beta_n^1)\|x_n - p\| + \delta\beta_n^1\|y_n^2 - p\| \\ &\le (1-\beta_n^1)\|x_n - p\| + \delta\beta_n^1[(1-\beta_n^2)\|x_n - p\| + \delta\beta_n^2\|y_n^3 - p\|] \\ &= [(1-\beta_n^1) + \delta\beta_n^1(1-\beta_n^2)]\|x_n - p\| + \delta^2\beta_n^1\beta_n^2\|y_n^3 - p\|. \end{aligned} \tag{2.22}$$

$$\|y_n^3 - p\| \le (1-\beta_n^3)\|x_n - p\| + \delta\beta_n^3\|y_n^4 - p\|. \tag{2.23}$$

$$\vdots$$

$$\|y_n^{k-2} - p\| \le (1-\beta_n^{k-2})\|x_n - p\| + \delta\beta_n^{k-2}\|y_n^{k-1} - p\|. \tag{2.24}$$

$$\begin{aligned} \|y_n^{k-1} - p\| &\le (1-\beta_n^{k-1})\|x_n - p\| + \delta\beta_n^{k-1}\|x_n - p\| \\ &= [1 - \beta_n^{k-1} + \delta\beta_n^{k-1}]\|x_n - p\|. \end{aligned} \tag{2.25}$$

Substituting (2.22) to (2.25) in (2.21) and simplifying, we have

$$\begin{aligned}
(2.26)\qquad \|x_{n+1}-p\| &\le \delta[1-(1-\delta^{k-1})]\alpha_n\beta_n^1\beta_n^2\beta_n^3\dots\beta_n^{k-2}\beta_n^{k-1}\|x_n-p\| \\
&\le \prod_{m=1}^{n}\delta[1-(1-\delta^{k-1})]\alpha_m\beta_m^1\beta_m^2\beta_m^3\dots\beta_m^{k-2}\beta_m^{k-1}\|x_1-p\|.
\end{aligned}$$

Applying the conditions $0\le\delta<1$, $\alpha_n,\beta_n^i\in(0,1)$ for $i=1,2,3,...,k-1$ in (2.26), it result that $\lim_{n\to\infty}\|x_{n+1}-p\|=0$. □

Theorem 2.2. *Let E be a Banach space, C a closed convex subset of E and $T:C\to C$ a selfmap of C with a fixed point p satisfying the condition*

$$(2.27)\qquad \|p-Ty\|\le\delta\|p-y\|,\ \forall x,y\in C,$$

for some $\delta\in[0,1)$. For $x_1\in C$, let $\{x_n\}_{n=1}^{\infty}$ be the modified multistep iterative scheme (B) defined by (2.18), where the sequences $\{\alpha_n\}_{n=1}^{\infty},\{\beta_n^i\}_{n=1}^{\infty}\subset(0,1)$, for $(i=1,2,...,k-1)$. Then $\{x_n\}_{n=1}^{\infty}$ (2.18) converges strongly to the fixed point p of T.

Proof. In view of (2.18) and (2.27), we have

$$\begin{aligned}
(2.28)\qquad \|x_{n+1}-p\| &= \|(1-\alpha_n)Tx_n+\alpha_nTy_n^1-p\| \\
&\le (1-\alpha_n)\|Tx_n-p\|+\alpha_n\|Ty_n^1-p\| \\
&\le \delta(1-\alpha_n)\|x_n-p\|+\delta\alpha_n\|y_n^1-p\|.
\end{aligned}$$

$$\begin{aligned}
(2.29)\qquad \|y_n^1-p\| &\le (1-\beta_n^1)\|y_n^2-p\|+\beta_n^1\|Ty_n^2-p\| \\
&= [1-\beta_n^1+\delta\beta_n^1]\|y_n^2-p\|.
\end{aligned}$$

$$(2.30)\qquad \|y_n^2-p\| \le [1-\beta_n^2+\delta\beta_n^2]\|y_n^3-p\|.$$

$$(2.31)\qquad \|y_n^3-p\| \le [1-\beta_n^3+\delta\beta_n^3]\|y_n^4-p\|.$$

$$\vdots$$

$$(2.32)\qquad \|y_n^{k-2}-p\| \le [1-\beta_n^{k-2}+\delta\beta_n^{k-2}]\|y_n^{k-1}-p\|.$$

$$\begin{aligned}
(2.33)\qquad \|y_n^{k-1}-p\| &\le (1-\beta_n^{k-1})\|x_n-p\|+\delta\beta_n^{k-1}\|x_n-p\| \\
&= [1-\beta_n^{k-1}+\delta\beta_n^{k-1}]\|x_n-p\|.
\end{aligned}$$

Substituting (2.29) to (2.33) in (2.28) and simplifying, we have

$$\begin{aligned}
(2.34)\qquad \|x_{n+1}-p\| &\le \delta[1-\alpha_n+\alpha_n[(1-(1-\delta)\beta_n^1)(1-(1-\delta)\beta_n^2) \\
&\quad (1-(1-\delta)\beta_n^3)\dots(1-(1-\delta)\beta_n^{k-2}) \\
&\quad (1-(1-\delta)\beta_n^{k-1})]]\|x_n-p\| \\
&\le \prod_{m=1}^{n}\delta[1-\alpha_m+\alpha_m[(1-(1-\delta)\beta_m^1)(1-(1-\delta)\beta_m^2) \\
&\quad (1-(1-\delta)\beta_m^3)\dots(1-(1-\delta)\beta_m^{k-2}) \\
&\quad (1-(1-\delta)\beta_m^{k-1})]]\|x_1-p\|.
\end{aligned}$$

Applying the conditions $0\le\delta<1$, $\alpha_n,\beta_n^i\in(0,1)$ for $i=1,2,3,...,k-1$ in (2.34), it result that $\lim_{n\to\infty}\|x_{n+1}-p\|=0$.
That is, $\{x_n\}_{n=1}^{\infty}$ in (2.18) converges strongly to the unique fixed point p of T. □

Theorem 2.3. *Let E be a Banach space, C a closed convex subset of E and $T : C \to C$ a self-map of C with a fixed point p satisfying the condition*

$$\|p - Ty\| \leq \delta\|p - y\|, \ \forall x, y \in C, \tag{2.35}$$

for some $\delta \in [0, 1)$. For $x_1 \in C$, let $\{x_n\}_{n=1}^{\infty}$ be the modified multistep iterative scheme (C) defined by (2.19), where the sequences $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n^i\}_{n=1}^{\infty} \subset (0, 1)$, for $(i = 1, 2, ..., k-1)$. Then $\{x_n\}_{n=1}^{\infty}$ (2.19) converges strongly to the fixed point p of T

Proof. In view of (2.19) and (2.35), we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1-\alpha_n)Ty_n^1 + \alpha_n Ty_n^1 - p\| \\ &\leq (1-\alpha_n)\|Ty_n^1 - p\| + \alpha_n\|Ty_n^1 - p\| \\ &\leq \delta(1-\alpha_n)\|y_n^1 - p\| + \delta\alpha_n\|y_n^1 - p\| \\ &= \delta\|y_n^1 - p\|. \end{aligned} \tag{2.36}$$

$$\begin{aligned} \|y_n^1 - p\| &\leq (1-\beta_n^1)\|y_-^2 p\| + \beta_n^1\|Ty_n^2 - p\| \\ &= [1 - \beta_n^1 + \delta\beta_n^1]\|y_n^2 - p\|. \end{aligned} \tag{2.37}$$

$$\|y_n^2 - p\| \leq [1 - \beta_n^2 + \delta\beta_n^2]\|y_n^3 - p\|. \tag{2.38}$$

$$\|y_n^3 - p\| \leq [1 - \beta_n^3 + \delta\beta_n^3]\|y_n^4 - p\|. \tag{2.39}$$

$$\vdots$$

$$\|y_n^{k-2} - p\| \leq [1 - \beta_n^{k-2} + \delta\beta_n^{k-2}]\|y_n^{k-1} - p\|. \tag{2.40}$$

$$\begin{aligned} \|y_n^{k-1} - p\| &\leq (1-\beta_n^{k-1})\|x_n - p\| + \delta\beta_n^{k-1}\|x_n - p\| \\ &= [1 - \beta_n^{k-1} + \delta\beta_n^{k-1}]\|x_n - p\|. \end{aligned} \tag{2.41}$$

Substituting (2.37) to (2.41) in (2.36) and simplifying, we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq \delta[1 - (1-\delta)\delta^{k-2})]\beta_n^1\beta_n^2\beta_n^3 \ldots \beta_n^{k-2}\beta_n^{k-1}\|x_n - p\| \\ &\leq \prod_{m=1}^{n} \delta[1 - (1-\delta)\delta^{k-2}]\beta_m^1\beta_m^2\beta_m^3 \ldots \beta_m^{k-2}\beta_m^{k-1}\|x_1 - p\|. \end{aligned} \tag{2.42}$$

Applying the conditions $0 \leq \delta < 1$, $\beta_n^i \in (0, 1)$ for $i = 1, 2, 3, ..., k-1$ in (2.42), it result that $\lim_{n\to\infty} \|x_{n+1} - p\| = 0$.

That is, $\{x_n\}_{n=1}^{\infty}$ in (2.19) converges strongly to the unique fixed point p of T. □

Theorem 2.4. *Let E be a Banach space, C a closed convex subset of E and $T : C \to C$ a self-map of C with a fixed point p satisfying the contractive condition (1.8) for some $\delta \in [0, 1)$. For $x_1 \in C$, let $\{x_n\}_{n=1}^{\infty}$ be the Thakur et al. iterative scheme defined by (1.16), where the sequences $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}\{\gamma_n\}_{n=1}^{\infty} \subset (0, 1)$. Then $\{x_n\}_{n=1}^{\infty}$ (1.16) converges strongly to the unique fixed point p of T.*

Proof. The method of proof of **Theorem 2.4** is similar to that of **Theorem 2.3**. The final result is

$$\begin{aligned} \|x_{n+1} - p\| &\leq \delta[1 - (1-\delta^2)]\alpha_n\beta_n\gamma_n\|x_n - p\| \\ &\leq \prod_{m=1}^{n} \delta[1 - (1-\delta^2)]\alpha_m\beta_m\gamma_m\|x_1 - p\|. \end{aligned} \tag{2.43}$$

Applying the conditions $0 \leq \delta < 1$, $\alpha_n\beta_n\gamma_n \in (0,1)$ in (2.43), it result that $\lim_{n\to\infty} \|x_{n+1} - p\| = 0$.
That is, $\{x_n\}_{n=1}^{\infty}$ in (1.16) converges strongly to the unique fixed point p of T. □

Theorem 2.5. *Let E be a Banach space, C a closed convex subset of E and $T : C \to C$ a self-map of C with a fixed point p satisfying the contractive condition (1.8) for some $\delta \in [0,1)$. For $x_1 \in C$, let $\{x_n\}_{n=1}^{\infty}$ be the Abbas et al. iterative scheme defined by (1.15), where the sequences $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}\{\gamma_n\}_{n=1}^{\infty} \subset (0,1)$. Then $\{x_n\}_{n=1}^{\infty}$ (1.15) converges strongly to the unique fixed point p of T.*

Proof. The method of proof of **Theorem 2.5** is similar to that of **Theorem 2.3**. The final result is

$$\begin{aligned} \|x_{n+1} - p\| &\leq \delta[1 - (1-\delta)]\alpha_n\beta_n\gamma_n\|x_n - p\| \\ &\leq \prod_{m=1}^{n} \delta[1 - (1-\delta)]\alpha_m\beta_m\gamma_m\|x_1 - p\|. \end{aligned} \tag{2.44}$$

Applying the conditions $0 \leq \delta < 1$, $\alpha_n, \beta_n, \gamma_n \in (0,1)$ in (2.44), it result that $\lim_{n\to\infty} \|x_{n+1} - p\| = 0$.
That is, $\{x_n\}_{n=1}^{\infty}$ in (1.15) converges strongly to the unique fixed point p of T. □

Theorem 2.6. *Let E be a Banach space, C a closed convex subset of E and $T : C \to C$ a self-map of C with a fixed point p satisfying the contractive condition (1.8) for some $\delta \in [0,1)$. For $x_1 \in C$, let $\{x_n\}_{n=1}^{\infty}$ be the Noor iterative scheme defined by (1.14), where the sequences $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}\{\gamma_n\}_{n=1}^{\infty} \subset (0,1)$. Then $\{x_n\}_{n=1}^{\infty}$ (1.14) converges strongly to the unique fixed point p of T.*

Proof. The method of proof of **Theorem 2.6** is similar to that of **Theorem 2.1**. The final result is

$$\begin{aligned} \|x_{n+1} - p\| &\leq [1 - (1-\delta)\alpha_n - (1-\delta)\delta\alpha_n\beta_n - (1-\delta)\delta^2\alpha_n\beta_n\gamma_n]\|x_n - p\| \\ &\quad \leq [1 - (1-\delta)\delta^2\alpha_n\beta_n\gamma_n]\|x_n - p\| \\ &\leq \prod_{m=1}^{n} [1 - (1-\delta)\delta^2\alpha_m\beta_m\gamma_m]\|x_1 - p\|. \end{aligned} \tag{2.45}$$

Applying the conditions $0 \leq \delta < 1$, $\alpha_n, \beta_n, \gamma_n \in (0,1)$ in (2.45), it result that $\lim_{n\to\infty} \|x_{n+1} - p\| = 0$.
That is, $\{x_n\}_{n=1}^{\infty}$ in (1.14) converges strongly to the unique fixed point p of T. □

Theorem 2.7. *Let E be a Banach space, C a closed convex subset of E and $T : C \to C$ a of C with a fixed point p satisfying the contractive condition (1.8) for some $\delta \in [0,1)$. For $x_1 \in C$, let $\{x_n\}_{n=1}^{\infty}$ be the Agarwal et al. iterative scheme defined by (1.13), where the sequences $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty} \subset (0,1)$. Then $\{x_n\}_{n=1}^{\infty}$ (1.13) converges strongly to the unique fixed point p of T*

Proof. The method of proof of **Theorem 2.7** is similar to that of **Theorem 2.2**. The final result is

$$\begin{aligned} \|x_{n+1} - p\| &\leq \delta[1 - (1-\delta)\alpha_n\beta_n]\|x_n - p\| \\ &\leq \prod_{m=1}^{n} \delta[1 - (1-\delta)\alpha_m\beta_m]\|x_1 - p\|. \end{aligned} \tag{2.46}$$

Applying the conditions $0 \leq \delta < 1$, $\alpha_n, \beta_n \in (0,1)$ in (2.46), it result that $\lim_{n\to\infty} \|x_{n+1} - p\| = 0$.

That is, $\{x_n\}_{n=1}^{\infty}$ in (1.13) converges strongly to the unique fixed point p of T. This ends the proof. $\square$

Theorem 2.8. *Let E be a Banach space, C a closed convex subset of E and $T : C \to C$ a self-map of C with a fixed point p satisfying the contractive condition (1.8) for some $\delta \in [0, 1)$. For $x_1 \in C$, let $\{x_n\}_{n=1}^{\infty}$ be the Ishikawa iterative scheme defined by (1.12), where the sequences $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty} \subset (0, 1)$. Then $\{x_n\}_{n=1}^{\infty}$ (1.12) converges strongly to the unique fixed point p of T.*

Proof. The method of proof of **Theorem 2.8** is similar to that of **Theorem 2.1** The final result is

$$\begin{aligned} \|x_{n+1} - p\| &\leq [1 - (1 - \delta)\alpha_n - (1 - \delta)\delta\alpha_n\beta_n]\|x_n - p\| \\ &\leq [1 - (1 - \delta)\delta\alpha_n\beta_n]\|x_n - p\| \\ &\leq \prod_{m=1}^{n} [1 - (1 - \delta)\delta\alpha_m\beta_m]\|x_1 - p\|. \end{aligned} \tag{2.47}$$

Applying the conditions $0 \leq \delta < 1, \alpha_n, \beta_n \in (0, 1)$ in (2.47), it result that $\lim_{n\to\infty} \|x_{n+1} - p\| = 0$.
That is, $\{x_n\}_{n=1}^{\infty}$ in (1.12) converges strongly to the unique fixed point p of T. $\square$

Theorem 2.9. *Let E be a Banach space, C a closed convex subset of E and $T : C \to C$ a self-map of C with a fixed point p satisfying the contractive condition (1.8) for some $\delta \in [0, 1)$. For $x_1 \in C$, let $\{x_n\}_{n=1}^{\infty}$ be the Picard-Mann hybrid iterative scheme defined by (1.11), where the sequence $\{\alpha_n\}_{n=1}^{\infty} \subset (0, 1)$. Then $\{x_n\}_{n=1}^{\infty}$ (1.11) converges strongly to the unique fixed point p of T.*

Proof. The method of proof of **Theorem 2.9** is similar to that of **Theorem 2.1**. The final result is

$$\begin{aligned} \|x_{n+1} - p\| &\leq \delta[1 - (1 - \delta)\alpha_n]\|x_n - p\| \\ &\leq \prod_{m=1}^{n} \delta[1 - (1 - \delta)\alpha_m]\|x_1 - p\|. \end{aligned} \tag{2.48}$$

Applying the conditions $0 \leq \delta < 1, \alpha_n \in (0, 1)$ in (2.48), it result that $\lim_{n\to\infty} \|x_{n+1} - p\| = 0$.
That is, $\{x_n\}_{n=1}^{\infty}$ in (1.11) converges strongly to the unique fixed point p of T. This ends the proof. $\square$

Theorem 2.10. *Let E be a Banach space, C a closed convex subset of E and $T : C \to C$ a self-map of C with a fixed point p satisfying the contractive condition (1.8) for some $\delta \in [0, 1)$. For $x_1 \in C$, let $\{x_n\}_{n=1}^{\infty}$ be the Mann iterative scheme defined by (1.10), where the sequence $\{\alpha_n\}_{n=1}^{\infty} \subset (0, 1)$. Then $\{x_n\}_{n=1}^{\infty}$ (1.10) converges strongly to the unique fixed point p of T.*

Proof. The method of proof of **Theorem 2.10** is similar to that of **Theorem 2.1**. The final result is

$$\begin{aligned} \|x_{n+1} - p\| &\leq [1 - (1 - \delta)\alpha_n]\|x_n - p\| \\ &\leq \prod_{m=1}^{n} [1 - (1 - \delta)\alpha_m]\|x_1 - p\|. \end{aligned} \tag{2.49}$$

Applying the conditions $0 \leq \delta < 1, \alpha_n \in (0, 1)$ in (2.49), it result that $\lim_{n\to\infty} \|x_{n+1} - p\| = 0$.
That is, $\{x_n\}_{n=1}^{\infty}$ in (1.10) converges strongly to the unique fixed point p of T. $\square$

Theorem 2.11. *Let E be a Banach space, C a closed convex subset of E and $T : C \to C$ a self-map of C with a fixed point p satisfying the contractive condition (1.8) for some $\delta \in [0, 1)$. For $x_1 \in C$, let $\{x_n\}_{n=1}^{\infty}$ be the Picard iterative scheme defined by (1.9). Then $\{x_n\}_{n=1}^{\infty}$ (1.9) converges strongly to the unique fixed point p of T.*

Proof. The method of proof of **Theorem 2.11** is similar to that of **Theorem 2.1**. The final result is

$$\|x_{n+1} - p\| \quad \leq \quad \delta\|x_n - p\|. \tag{2.50}$$

Applying the condition $0 \leq \delta < 1$ in (2.50), it result that $\lim_{n\to\infty} \|x_{n+1} - p\| = 0$.
That is, $\{x_n\}_{n=1}^{\infty}$ in (1.9 converges strongly to the unique fixed point p of T. □

3. Main Result II

3.1. **Convergence Results for Non-expansive Mapping in Uniformly Convex Banach Spaces.** In this section, we present some convergence fixed point results for the class of non-expansive mappings in uniformly convex Banach spaces.
We now prove our main results II using iterative schemes (2.17), (2.18) and (2.19) as follows:

Theorem 3.12. *Let E be a uniformly convex Banach space, C a non-empty closed convex subset of E and $T : C \to C$ a non-expansive self mapping of C satisfying the condition*

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in C. \tag{3.51}$$

For $x_1 \in C$, let $\{x_n\}_{n=1}^{\infty}$ be the modified multistep iterative scheme (A) defined by (2.17), where the sequences $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n^i\}_{n=1}^{\infty} \subset [\epsilon, 1-\epsilon]$, for some $\epsilon \in (0, 1)$, where $(i = 1, 2, ..., k-1)$. If $F_T \neq \emptyset$, then $lim_{n\to\infty}\|x_n - Tx_n\| = 0$.

Proof. Suppose $p \in F_T$, then applying (3.51) on (2.17), we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1-\alpha_n)Tx_n + \alpha_n Ty_n^1 - p\| \\ &\leq (1-\alpha_n)\|Tx_n - p\| + \alpha_n\|Ty_n^1 - p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|y_n^1 - p\|. \end{aligned} \tag{3.52}$$

$$\begin{aligned} \|y_n^1 - p\| &\leq (1-\beta_n^1)\|x_n - p\| + \beta_n^1\|Ty_n^2 - p\| \\ &\leq (1-\beta_n^1)\|x_n - p\| + \beta_n^1\|y_n^2 - p\|. \end{aligned} \tag{3.53}$$

$$\|y_n^2 - p\| \leq (1-\beta_n^2)\|x_n - p\| + \beta_n^2\|y_n^3 - p\|. \tag{3.54}$$

$$\|y_n^3 - p\| \leq (1-\beta_n^3)\|x_n - p\| + \beta_n^3\|y_n^4 - p\|. \tag{3.55}$$

$$\vdots$$

$$\|y_n^{k-2} - p\| \leq (1-\beta_n^{k-2})\|x_n - p\| + \beta_n^{k-2}\|y_n^{k-1} - p\|. \tag{3.56}$$

$$\begin{aligned} \|y_n^{k-1} - p\| &\leq (1-\beta_n^{k-1})\|x_n - p\| + \beta_n^{k-1}\|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{3.57}$$

Substituting (3.53) to (3.57) in (3.52) and simplifying, we have

$$\|x_{n+1} - p\| \quad \leq \quad \|x_n - p\|. \tag{3.58}$$

Thus, $\lim_{n\to\infty} \|x_{n+1} - p\|$ exists. Let $\lim_{n\to\infty} \|x_{n+1} - p\| = c$.
Now,

(3.59) $$\|y_n^{k-1}-p\| \leq \|x_n-p\| \Rightarrow lim_{n\to\infty} sup\|y_n^{k-1}-p\| \quad \leq \quad c.$$

(3.60) $$\|y_n^{k-2}-p\| \leq \|x_n-p\| \Rightarrow lim_{n\to\infty} sup\|y_n^{k-2}-p\| \quad \leq \quad c.$$

(3.61) $$\|y_n^{3}-p\| \leq \|x_n-p\| \Rightarrow lim_{n\to\infty} sup\|y_n^{3}-p\| \quad \leq \quad c.$$

(3.62) $$\|y_n^{2}-p\| \leq \|x_n-p\| \Rightarrow lim_{n\to\infty} sup\|y_n^{2}-p\| \quad \leq \quad c.$$

(3.63) $$\|y_n^{1}-p\| \leq \|x_n-p\| \Rightarrow lim_{n\to\infty} sup\|y_n^{1}-p\| \quad \leq \quad c.$$

$c = \lim_{n\to\infty} \|x_{n+1}-p\| = \lim_{n\to\infty} \|(1-\alpha_n)(Tx_n-p)+\alpha_n(Ty_n^1-p)\|$ holds for some $c \geq 0$.
Thus, by Lemma 1.2,

(3.64) $$\lim_{n\to\infty} \|(1-\alpha_n)(Tx_n-Ty_n^1)\| = 0.$$

Therefore,

$$\|x_{n+1}-p\| = \|(Tx_n-p)+\alpha_n(Tx_n-Ty_n^1)\| \leq \|(Tx_n-p)\| + \alpha_n\|(Tx_n-Ty_n^1)\|,$$

yields

(3.65) $$lim_{n\to\infty} inf\|Tx_n-p\|.$$

$$\|Tx_n-p\| \leq \|Tx_n-Ty_n^1\| + \|Ty_n^1-p\| \leq \|Tx_n-Ty_n^1\| + \|y_n^1-p\|.$$

Thus,

(3.66) $$lim_{n\to\infty} inf\|y_n^1-p\|.$$

From (3.63) and (3.66), we obtain

(3.67) $$lim_{n\to\infty}\|y_n^1-p\| = c.$$

Also,

$$\|Ty_n^1-p\| \leq \|Ty_n^1-Ty_n^2\| + \|Ty_n^2-p\| \leq \|Ty_n^1-Ty_n^2\| + \|y_n^2-p\|.$$

Thus,

(3.68) $$lim_{n\to\infty} inf\|y_n^2-p\|.$$

From (3.62) and (3.68),we obtain

(3.69) $$lim_{n\to\infty}\|y_n^2-p\| = c.$$

Similarly,

$$\|Ty_n^2-p\| \leq \|Ty_n^2-Ty_n^3\| + \|Ty_n^3-p\| \leq \|Ty_n^2-Ty_n^3\| + \|y_n^3-p\|.$$

Thus,

(3.70) $$lim_{n\to\infty} inf\|y_n^3-p\|.$$

From (3.61) and (3.70),we obtain

(3.71) $$lim_{n\to\infty}\|y_n^3-p\| = c.$$

$$\|Ty_n^{k-2}-p\| \leq \|Ty_n^{k-2}-Ty_n^{k-1}\| + \|Ty_n^{k-1}-p\| \leq \|Ty_n^{k-2}-Ty_n^{k-1}\| + \|y_n^{k-1}-p\|.$$

Thus,

(3.72) $$lim_{n\to\infty} inf\|y_n^{k-1}-p\|.$$

From (3.59) and (3.72),we obtain

(3.73) $$lim_{n\to\infty}\|y_n^{k-1}-p\| = c.$$

Therefore,

$$c = \lim_{n\to\infty} \|y_n^{k-1} - p\| = \lim_{n\to\infty} \|(1-\beta_n^{k-1})(x_n - p) + \beta_n^{k-1}(Tx_n - p)\|$$

and by Lemma 1.3, we have the result $\lim_{n\to\infty} \|Tx_n - x_n\|$. □

Theorem 3.13. *Let E be a uniformly convex Banach space, C a non-empty closed convex subset of E and $T : C \to C$ a non-expansive self mapping of C satisfying the condition*

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in C. \tag{3.74}$$

For $x_1 \in C$, let $\{x_n\}_{n=1}^{\infty}$ be the modified multistep iterative scheme (B) defined by (2.18), where the sequences $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n^i\}_{n=1}^{\infty} \subset [\epsilon, 1-\epsilon]$, for some $\epsilon \in (0,1)$, where $(i = 1, 2, ..., k-1)$. If $F_T \neq \emptyset$, then $lim_{n\to\infty}\|x_n - Tx_n\| = 0$.

Proof. The method of proof of **Theorem 3.13** is similar to that of **Theorem 3.12**. □

Theorem 3.14. *Let E be a uniformly convex Banach space, C a non-empty closed convex subset of E and $T : C \to C$ a non-expansive self mapping of C satisfying the condition*

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in C. \tag{3.75}$$

For $x_1 \in C$, let $\{x_n\}_{n=1}^{\infty}$ be the modified multistep iterative scheme (C) defined by (2.19), where the sequences $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n^i\}_{n=1}^{\infty} \subset [\epsilon, 1-\epsilon]$, for some $\epsilon \in (0,1)$, where $(i = 1, 2, ..., k-1)$. If $F_T \neq \emptyset$, then $lim_{n\to\infty}\|x_n - Tx_n\| = 0$.

Proof. The method of proof of **Theorem 3.14** is similar to that of **Theorem 3.12** . □

Theorem 3.15. *Let E be a uniformly convex Banach space, C a non-empty closed convex subset of E and $T : C \to C$ a non-expansive self mapping of C satisfying the condition*

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in C. \tag{3.76}$$

For $x_1 \in C$, let $\{x_n\}_{n=1}^{\infty}$ be the iterative schemes defined respectively by (1.14), (1.13), (1.12), (1.11), (1.10) and (1.9), where the sequences $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}, \{\gamma_n\}_{n=1}^{\infty} \subset [\epsilon, 1-\epsilon]$, for some $\epsilon \in (0,1)$. If $F_T \neq \emptyset$, then $lim_{n\to\infty}\|x_n - Tx_n\| = 0$.

Proof. The method of proof of **Theorem 3.15** is similar to that of **Theorem 3.12**. □

4. MAIN RESULT III

4.1. **Application of Modified Multistep Iterative Schemes to Constrained Optimization and Split Feasibility Problems in Real Hilbert Spaces.** In this section, we give a brief explanation of constrained optimization and split feasibility problems in the framework of Hilbert spaces, we will also present the relevance of existence and approximation of solutions in variational inequalities and prove some useful results using the modified multistep iterative schemes (A1), (B1) and (C1) for non-expansive mappings.

Let H be a Hilbert space and C be a non-empty, closed subset of H. $T : C \to H$ be a nonself operator.
Let $VI(C,T)$ represent the variational inequality problem defined by C and T. Let $\Omega(C,T)$ be the set of all vectors which solves $VI(C,T)$ problem. Fixed point problem has an equivalent relationship with $VI(C,T)$ problem in the sense that $x^* = F_\mu x^* = P_C(I - \mu T)x^*$, where $x^* \in C$, $\mu > 0$ is (a constant), P_C is the metric projection from H onto C and $F_\mu = P_C(I - \mu T)$.
The set of all fixed points in the $VI(C,T)$ is defined by

$$\Omega(C,T) = \{x^* : x^* = F_\mu x^*\} = \{x^* : x^* = P_C(I - \mu T)x^*\}.$$

We now prove the following theorems that deal with variational inequality problem.

Definition 4.5. *Let H be a Hilbert space and C be a non-empty, closed subset of H. $T : C \to H$ be a nonself operator. T is called a contraction if*

$$< P_C(I - \mu T)x, P_C(I - \mu T)y > \leq \mu < x, y >, \ \forall x, y \in C, \tag{4.77}$$

where $0 < \mu < \frac{2\lambda}{L^2}$, $L > 0$.

Definition 4.6. *Let H be a Hilbert space and C be a non-empty, closed subset of H. Let $T : C \to H$ be an $L-$ Lipschitzian and $\lambda-$ strongly monotone operator with $\mu \in (0, \frac{2\mu}{L^2})$. Let $\Omega(C, T)$ be the set of all fixed points in the $VI(C, T)$ problem. The modified multistep iterative scheme $(A1)$ is the sequence $\{x_n\}_{n=1}^{\infty} \subset C$ defined by:*

$$\begin{aligned} & x_1 \in C, \\ & x_{n+1} = (1 - \alpha_n)P_C(I - \mu T)x_n + \alpha_n P_C(I - \mu T)y_n^1, \\ & y_n^i = (1 - \beta_n^i)x_n + \beta_n^i P_C(I - \mu T)y_n^{i+1}, \ i = 1, 2, \ldots, k - 2, \\ & y_n^{k-1} = (1 - \beta_n^{k-1})x_n + \beta_n^{k-1} P_C(I - \mu T)x_n, \ k = 2, 3, \ldots, \ n = 1, 2, \ldots, \end{aligned} \tag{4.78}$$

where the sequences $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n^i\}_{n=1}^{\infty} \subset [\epsilon, 1 - \epsilon]$ for some $\epsilon \in (0, 1)$, for $(i = 1, 2, ..., k - 1)$.

Definition 4.7. *Let H be a Hilbert space and C be a nonempty, closed subset of H. Let $T : C \to H$ be an $L-$ Lipschitzian and $\lambda-$ strongly monotone operator with $\mu \in (0, \frac{2\mu}{L^2})$. Let $\Omega(C, T)$ be the set of all fixed points in the $VI(C, T)$ problem. The modified multistep iterative scheme $(B1)$ is the sequence $\{x_n\}_{n=1}^{\infty} \subset C$ defined by:*

$$\begin{aligned} & x_1 \in C, \\ & x_{n+1} = (1 - \alpha_n)P_C(I - \mu T)x_n + \alpha_n P_C(I - \mu T)y_n^1, \\ & y_n^i = (1 - \beta_n^i)y_n^{i+1} + \beta_n^i P_C(I - \mu T)y_n^{i+1}, \ i = 1, 2, \ldots, k - 2, \\ & y_n^{k-1} = (1 - \beta_n^{k-1})x_n + \beta_n^{k-1} P_C(I - \mu T)x_n, \ k = 2, 3, \ldots, \ n = 1, 2, \ldots, \end{aligned} \tag{4.79}$$

where the sequences $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n^i\}_{n=1}^{\infty} \subset [\epsilon, 1 - \epsilon]$ for some $\epsilon \in (0, 1)$, for $(i = 1, 2, ..., k - 1)$.

Definition 4.8. *Let H be a Hilbert space and C be a nonempty, closed subset of H. Let $T : C \to H$ be an $L-$ Lipschitzian and $\lambda-$ strongly monotone operator with $\mu \in (0, \frac{2\mu}{L^2})$. Let $\Omega(C, T)$ be the set of all fixed points in the $VI(C, T)$ problem. The modified multistep iterative scheme $(C1)$ is the sequence $\{x_n\}_{n=1}^{\infty} \subset C$ defined by:*

$$\begin{aligned} & x_1 \in C, \\ & x_{n+1} = (1 - \alpha_n)P_C(I - \mu T)y_n^1 + \alpha_n P_C(I - \mu T)y_n^1, \\ & y_n^i = (1 - \beta_n^i)y_n^{i+1} + \beta_n^i P_C(I - \mu T)y_n^{i+1}, \ i = 1, 2, \ldots, k - 2, \\ & y_n^{k-1} = (1 - \beta_n^{k-1})x_n + \beta_n^{k-1} P_C(I - \mu T)x_n, \ k = 2, 3, \ldots, \ n = 1, 2, \ldots, \end{aligned} \tag{4.80}$$

where the sequences $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n^i\}_{n=1}^{\infty} \subset [\epsilon, 1 - \epsilon]$ for some $\epsilon \in (0, 1)$, for $(i = 1, 2, ..., k - 1)$.

We now prove the following theorems that deal with variational inequality problem.

Theorem 4.16. *Let H be a Hilbert space and C be a nonempty, closed subset of H. Let $T : C \to H$ be an $L-$ Lipschitzian and $\lambda-$ strongly monotone operator with $\mu \in (0, \frac{2\mu}{L^2})$. Let $\Omega(C, T)$ be the set of all fixed points in the $VI(C, T)$ problem. Suppose $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n^i\}_{n=1}^{\infty} \subset [\epsilon, 1 - \epsilon]$ for some $\epsilon \in (0, 1)$, for $(i = 1, 2, ..., k - 1)$. For $x_1 \in C$, let $\{x_n\}_{n=1}^{\infty} \subset C$ be the modified multistep iterative schemes $(A1)$, $(B1)$ and $(C1)$ defined respectively by (4.78), (4.79) and (4.80). Then,*

(i) the modified multistep iterative scheme (4.78) converges weakly to $x^* \in \Omega(C,T)$;
(ii) the modified multistep iterative scheme (4.79) converges weakly to $x^* \in \Omega(C,T)$;
(iii) the modified multistep iterative scheme (4.80) converges weakly to $x^* \in \Omega(C,T)$.

4.2. **Application of Modified Multistep Iterative Schemes to Constrained Optimization Problems in Hilbert Spaces.** Iterative constrained optimization processes designed to minimize a convex differentiable function T over a closed convex set C in a Hilbert space are usually the algorithms for signal and image processing.

Let H be a Hilbert space and C be a nonempty, closed subset of H, P_C the metric projection of H onto C and let $T : C \to H$ be a $v - ism$ where $v > 0$ is a constant. Then $P_C(I - \mu T)$ is a nonexpansive operator provided $\mu \in (0, 2v)$.
The modified multistep iterative scheme $(A11)$ is the sequence $\{x_n\}_{n=1}^{\infty} \subset C$ defined by:

$$\begin{aligned}
&x_1 \in C,\\
&x_{n+1} = (1-\alpha_n)P_C(I-\bigtriangledown T)x_n + \alpha_n P_C(I-\bigtriangledown T)y_n^1,\\
&y_n^i = (1-\beta_n^i)x_n + \beta_n^i P_C(I-\bigtriangledown T)y_n^{i+1},\ i=1,2,\ldots,k-2,\\
&y_n^{k-1} = (1-\beta_n^{k-1})x_n + \beta_n^{k-1} P_C(I-\bigtriangledown T)x_n,\ k=2,3,\ldots,\ n=1,2,\ldots,
\end{aligned} \tag{4.81}$$

where the sequences $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n^i\}_{n=1}^{\infty} \subset [\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$, for $(i = 1,2,...,k-1)$.

The modified multistep iterative scheme $(B11)$ is the sequence $\{x_n\}_{n=1}^{\infty} \subset C$ defined by:

$$\begin{aligned}
&x_1 \in C,\\
&x_{n+1} = (1-\alpha_n)P_C(I-\bigtriangledown T)x_n + \alpha_n P_C(I-\bigtriangledown T)y_n^1,\\
&y_n^i = (1-\beta_n^i)y_n^{i+1} + \beta_n^i P_C(I-\bigtriangledown T)y_n^{i+1},\ i=1,2,\ldots,k-2,\\
&y_n^{k-1} = (1-\beta_n^{k-1})x_n + \beta_n^{k-1} P_C(I-\bigtriangledown T)x_n,\ k=2,3,\ldots,\ n=1,2,\ldots,
\end{aligned} \tag{4.82}$$

where the sequences $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n^i\}_{n=1}^{\infty} \subset [\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$, for $(i = 1,2,...,k-1)$.

The modified multistep iterative scheme $(C11)$ is the sequence $\{x_n\}_{n=1}^{\infty} \subset C$ defined by:

$$\begin{aligned}
&x_1 \in C,\\
&x_{n+1} = (1-\alpha_n)P_C(I-\bigtriangledown T)y_n^1 + \alpha_n P_C(I-\bigtriangledown T)y_n^1,\\
&y_n^i = (1-\beta_n^i)y_n^{i+1} + \beta_n^i P_C(I-\bigtriangledown T)y_n^{i+1},\ i=1,2,\ldots,k-2,\\
&y_n^{k-1} = (1-\beta_n^{k-1})x_n + \beta_n^{k-1} P_C(I-\bigtriangledown T)x_n,\ k=2,3,\ldots,\ n=1,2,\ldots,
\end{aligned} \tag{4.83}$$

where the sequences $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n^i\}_{n=1}^{\infty} \subset [\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$, for $(i = 1,2,...,k-1)$.
We shall now present results which generates the sequence of vectors in the constrained set C which converges weakly to the optimal solution which minimizes T as follows:

Theorem 4.17. *Let H be a Hilbert space and C be a nonempty, closed subset of H. Let T be a convex and differentiable function on an open set D containing the set C. Assume $\bigtriangledown T$ is an $L-$ Lipschitz operator on D with $\mu \in (0, \frac{2}{L})$ and there exist minimizers of T relative to the set C. Suppose $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n^i\}_{n=1}^{\infty} \subset [\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$, for $(i = 1,2,...,k-1)$. For $x_1 \in C$, let $\{x_n\}_{n=1}^{\infty} \subset C$ be the modified multistep iterative schemes $(A11)$, $(B11)$ and $(C11)$ defined respectively by (4.81), (4.82) and (4.83). Then,*
(i) the modified multistep iterative scheme (4.81) converges weakly to a minimizer of T;
(ii) the modified multistep iterative scheme (4.82) converges weakly to a minimizer of T;
(iii) the modified multistep iterative scheme (4.83) converges weakly to a minimizer of T.

4.3. **Application of Modified Multistep Iterative Schemes to Split Feasibility Problems in Hilbert Spaces.** Let $SFP(C,T)$ represent split feasibility problem of C and T. The $SFP(C,T)$ is to find a point

$$x \in C : \; Tx \in Q, \tag{4.84}$$

where C and Q are closed convex subsets of Hilbert spaces H_1 and H_2. The $SFP(C,T)$ is said to be consistent if (4.84) has a solution.
Find

$$x \in C : \; x = P_C(I - \gamma T^*(I - P_Q)T)x, \tag{4.85}$$

where P_C and P_Q are the orthogonal projections onto C and Q respectively with $\gamma > 0, T^*$ is the adjoint of T. For sufficiently small $\gamma > 0$ the operator $P_C(I - \gamma T^*(I - P_Q)T)$ is non-expansive. The $SFP(C,T)$ is said to be consistent if and only if the fixed point problem in (4.85) has a solution $x \in C$. Iterative constrained optimization processes designed to minimize a convex differentiable function T over a closed convex set C in a Hilbert space are usually the algorithms for signal and image processing.

Consider the minimization problem

$$min \; q(x)_{x \in C}. \tag{4.86}$$

where $q(x) = \frac{1}{2}\|(T - P_Q T)x\|, \forall x \in C$, then the gradient of q is $\bigtriangledown q = T^*(I - P_Q)T$, where T^* is the adjoint of T.
$\bigtriangledown q$ is an $L-$Lipschitzian with $L = \|T\|^2$ since $I - P_Q$ is nonexpansive. Thus, $\bigtriangledown q$ is $\frac{1}{L} - ism$ for any $\mu \in (0, \frac{2}{L})$ $I - \bigtriangledown q$ is averaged. Therefore the composition $P_C(I - \mu \bigtriangledown q) = T$ is averaged and the solution set of $SFP(C,T) = F(T)$.

The modified multistep iterative scheme $(A111)$ is the sequence $\{x_n\}_{n=1}^{\infty} \subset C$ defined by:

$$\begin{aligned} & x_1 \in C, \\ & x_{n+1} = (1 - \alpha_n)P_C(I - \mu \bigtriangledown q)x_n + \alpha_n P_C(I - \mu \bigtriangledown q)y_n^1, \\ & y_n^i = (1 - \beta_n^i)x_n + \beta_n^i P_C(I - \mu \bigtriangledown q)y_n^{i+1}, \; i = 1, 2, \ldots, k-2, \\ & y_n^{k-1} = (1 - \beta_n^{k-1})x_n + \beta_n^{k-1} P_C(I - \mu \bigtriangledown q)x_n, \; k = 2, 3, \ldots, \; n = 1, 2, \ldots, \end{aligned} \tag{4.87}$$

where the sequences $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n^i\}_{n=1}^{\infty} \subset [\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$, for $(i = 1, 2, ..., k-1)$ and $\mu \in (0, \frac{2}{\|T\|^2})$.

The modified multistep iterative scheme $(B111)$ is the sequence $\{x_n\}_{n=1}^{\infty} \subset C$ defined by:

$$\begin{aligned} & x_1 \in C, \\ & x_{n+1} = (1 - \alpha_n)P_C(I - \mu \bigtriangledown q)x_n + \alpha_n P_C(I - \mu \bigtriangledown q)y_n^1, \\ & y_n^i = (1 - \beta_n^i)y_n^{i+1} + \beta_n^i P_C(I - \mu \bigtriangledown q)y_n^{i+1}, \; i = 1, 2, \ldots, k-2, \\ & y_n^{k-1} = (1 - \beta_n^{k-1})x_n + \beta_n^{k-1} P_C(I - \mu \bigtriangledown q)x_n, \; k = 2, 3, \ldots, \; n = 1, 2, \ldots, \end{aligned} \tag{4.88}$$

where the sequences $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n^i\}_{n=1}^{\infty} \subset [\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$, for $(i = 1, 2, ..., k-1)$ and $\mu \in (0, \frac{2}{\|T\|^2})$.

The modified multistep iterative scheme $(C111)$ is the sequence $\{x_n\}_{n=1}^{\infty} \subset C$ defined by:

$$(4.89) \qquad \begin{aligned} & x_1 \in C, \\ & x_{n+1} = (1-\alpha_n)P_C(I-\mu \bigtriangledown q)y_n^1 + \alpha_n P_C(I-\mu \bigtriangledown q)y_n^1, \\ & y_n^i = (1-\beta_n^i)y_n^{i+1} + \beta_n^i P_C(I-\mu \bigtriangledown q)y_n^{i+1}, \ i = 1,2,\ldots,k-2, \\ & y_n^{k-1} = (1-\beta_n^{k-1})x_n + \beta_n^{k-1} P_C(I-\mu \bigtriangledown q)x_n, \ k = 2,3,\ldots, \ n = 1,2,\ldots, \end{aligned}$$

where the sequences $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n^i\}_{n=1}^{\infty} \subset [\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$, for $(i = 1,2,...,k-1)$ and $\mu \in (0, \frac{2}{\|T\|^2})$.

Theorem 4.18. [1] *Let E be a uniformly convex Banach space and let C be a nonempty closed convex subset E, T be a nonexpansive self mappings of C and $\{x_n\}_{n=1}^{\infty}$ be defined by the iterative scheme (1.15). Assume that (a) E satisfies Opial"s condition or (b) E has a Frechet differentiable norm. If $F(T) \neq \emptyset$, then $\{x_n\}_{n=1}^{\infty}$ converges weakly to a fixed point of T.*

We now use the iterative schemes (4.87) to (4.89) to find the solution of $SFP(C,T)$ in the following theorem:

Theorem 4.19. *Suppose that $SFP(C,T)$ is consistent. Let $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n^i\}_{n=1}^{\infty} \subset [\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$, for $(i = 1,2,...,k-1)$, where $\mu \in (0, \frac{2}{\|T\|^2})$. For $x_1 \in C$, let $\{x_n\}_{n=1}^{\infty} \subset C$ be the modified multistep iterative schemes $(A111)$, $(B111)$ and $(C111)$ defined respectively by (4.87), (4.88) and (4.89). Then,*
(i) the modified multistep iterative scheme (4.87) converges weakly to a solution of $SFP(C,T)$;
(ii) the modified multistep iterative scheme (4.88) converges weakly to a solution of $SFP(C,T)$;
(iii) the modified multistep iterative scheme (4.89) converges weakly to a solution of $SFP(C,T)$.

Proof. Since $T = P_C(I-\mu \bigtriangledown q)$ is nonexpansive, let $p \in SFP(C,T)$. Then, $\lim_{n\to\infty} \|x_n - p\|$ exists. We prove that $\{x_n\}$ has a unique weak subsequential limit in $SFP(C,T)$. Let u and v be weak limits of the subsequence $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ respectively.
By **Theorem 3.1**, $\lim_{n\to\infty} \|x_n - P_C(I-\mu \bigtriangledown q)x_n\| = 0$ and $I - T = I - P_C(I-\mu \bigtriangledown q)$ is demiclosed with respect to zero, thus we obtain $Tu = P_C(I-\mu \bigtriangledown q)u = u$. Similarly, we can prove that $v \in SFP(C,T)$.
Next, we prove uniqueness. Assume $u \neq v$, then by Opial condition,
$\lim_{n\to\infty} \|x_n - u\| = \lim_{n_i\to\infty} \|x_{n_i} - u\| < \lim_{n_i\to\infty} \|x_{n_i} - v\| = \lim_{n\to\infty} \|x_n - v\| = \lim_{n_j\to\infty} \|x_{n_j} - v\| < \lim_{n_j\to\infty} \|x_{n_j} - u\| = \lim_{n\to\infty} \|x_n - u\|$. This is a contradiction, so $u = v$.
Assume E has a Frechet differentiable norm, by Lemma 1.2, $< p - q, J(p_1 - p_2 >= 0$ for $p, q \in w_w(x_n)$. Therefore, $\|u - v\|^2 =< u - v, J(u - v >= 0$ implies $u = v$. Thus, (4.87) converges weakly to a solution of $SFP(C,T)$. This ends the proof.
(ii) The proof of (ii) is similar to that of (i). □

4.4. **Convergence Speed of the Various Iterative Schemes in Banach Spaces.** In this section, we present the convergence speed of iterative schemes (1.9) - (1.16), (2.17) - (2.19).
Let PMann represent Picard-Mann hybrid Iterative scheme (1.11);
Let $Agal\ IS$ represent Agarwal et al. iterative scheme (1.13);
Let AIS represent Abbas and Nazir iterative scheme (1.15);
Let TIS represent Thakur et al. iterative scheme (1.16);
Let $Mmultistep\ IS\ A$ represent modified multistep iterative scheme A (2.17);
Let $Mmultistep\ IS\ B$ represent modified multistep iterative scheme B (2.18);
Let $Mmultistep\ IS\ C$ represent modified multistep iterative scheme C (2.19).

Theorem 4.20. *Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two non-negative real sequences which converge to a and b respectively. Let $J = lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|}$,*
(i). if $J = 0$, then $\{a_n\}_{n=0}^{\infty}$ converges to a faster than $\{b_n\}_{n=0}^{\infty}$ converges to b;
(ii). if $0 < J < \infty$, then both $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ have the same convergence rate;
(iii). if $J = \infty$, then $\{b_n\}_{n=0}^{\infty}$ converges to b faster than $\{a_n\}_{n=0}^{\infty}$ converges to a.

Case 1. *Comparison of $Mmultistep\ IS\ (C)$ (2.19) and TIS (1.16) gives:*
Let $J = lim_{n\to\infty} \frac{\|x_{n+1}(M-multiIS(C)) - p\|}{\|x_{n+1}(TIS) - p\|}$, then

$$J \quad = \quad lim_{n\to\infty} \frac{\delta^n[(1-(1-\delta)\delta^{k-2})\beta^1\beta^2\beta^3 \ldots \beta^{k-2}\beta^{k-1}]^n \|x_1(MmultiIS)(C) - p\|}{\delta^n[(1-(1-\delta^2))\alpha\beta^1\beta^2]^n \|x_1(TIS) - p\|} = 0.$$

Thus, the modified multistep iterative scheme (C) (2.19) converges to the fixed point p faster than Thakur et al. iterative scheme (1.16) to p.

Case 2. *Comparison of $Mmultistep\ IS\ (C)$ (2.19) and AIS (1.15) gives:*
Let $J = lim_{n\to\infty} \frac{\|x_{n+1}(M-multiIS(C)) - p\|}{\|x_{n+1}(AIS) - p\|}$, then

$$J \quad = \quad lim_{n\to\infty} \frac{\delta^n[(1-(1-\delta)\delta^{k-2})\beta^1\beta^2\beta^3 \ldots \beta^{k-2}\beta^{k-1}]^n \|x_1(MmultiIS)(C) - p\|}{\delta^n[(1-(1-\delta))\alpha\beta^1\beta^2]^n \|x_1(AIS) - p\|} = 0.$$

Thus, the modified multistep iterative scheme (C) (2.19) converges to the fixed point p faster than Abbas and Nazir iterative scheme (1.15) to p.

Case 3.*Comparison of $Mmultistep\ IS\ (C)$ (2.19) and Noor (1.14) gives:*
Let $J = lim_{n\to\infty} \frac{\|x_{n+1}(M-multiIS(C)) - p\|}{\|x_{n+1}(Noor) - p\|}$, then

$$J \quad = \quad lim_{n\to\infty} \frac{\delta^n[(1-(1-\delta)\delta^{k-2})\beta^1\beta^2\beta^3 \ldots \beta^{k-2}\beta^{k-1}]^n \|x_1(Mmulti\ IS)(C) - p\|}{[(1-(1-\delta)\delta^2)\alpha\beta^1\beta^2]^n \|x_1(Noor) - p\|} = 0.$$

Thus, the modified multistep iterative scheme (C) (2.19) converges to the fixed point p faster than Noor iterative scheme (1.14) to p.

Case 4. *Comparison of $Mmultistep\ IS\ (C)$ (2.19) and $AgarIS$ (1.13) gives:*
Let $J = lim_{n\to\infty} \frac{\|x_{n+1}(M-multiIS(C)) - p\|}{\|x_{n+1}(AgarIS) - p\|}$, then

$$J \quad = \quad lim_{n\to\infty} \frac{\delta^n[(1-(1-\delta)\delta^{k-2})\beta^1\beta^2\beta^3 \ldots \beta^{k-2}\beta^{k-1}]^n \|x_1(Mmulti\ IS)(C) - p\|}{\delta^n[(1-(1-\delta))\alpha\beta^1]^n \|x_1(AgarIS) - p\|} = 0.$$

Thus, the modified multistep iterative scheme (C) (2.19) converges to the fixed point p faster than Agarwal et al. iterative scheme (1.13) to p.

Case 5. *Comparison of $Mmultistep\ IS\ (C)$ (2.19) and Ishikawa (1.12) gives:*
Let $J = lim_{n\to\infty} \frac{\|x_{n+1}(M-multiIS(C)) - p\|}{\|x_{n+1}(Ishi) - p\|}$, then

$$J \quad = \quad lim_{n\to\infty} \frac{\delta^n[(1-(1-\delta)\delta^{k-2})\beta^1\beta^2\beta^3 \ldots \beta^{k-2}\beta^{k-1}]^n \|x_1(Mmulti\ IS)(C) - p\|}{[(1-(1-\delta)\delta^2)\alpha\beta^1]^n \|x_1(Ishi) - p\|} = 0.$$

Thus, the modified multistep iterative scheme (C) (2.19) converges to the fixed point p faster than Ishikawa iterative scheme (1.12) to p.

Case 6. *Comparison of $Mmultistep\ IS\ (C)$ (2.19) and $PMann\ IS$ (1.11) gives:*

Let $J = lim_{n\to\infty} \frac{\|x_{n+1}(M-multiIS(C))-p\|}{\|x_{n+1}(PMann)-p\|}$*, then*

$$J \quad = \quad lim_{n\to\infty} \frac{\delta^n[(1-(1-\delta)\delta^{k-2})\beta^1\beta^2\beta^3\ldots\beta^{k-2}\beta^{k-1}]^n\|x_1(Mmulti\ IS)(C)-p\|}{\delta^n[(1-(1-\delta))\alpha]^n\|x_1(PMann)-p\|} = \infty.$$

Thus, the modified multistep iterative scheme (C) *(2.19) converges to the fixed point* p *faster than Picard-Mann iterative scheme (1.11)* p*.*

Case 7. *Comparison of* $Mmultistep\ IS$ (C) *(2.19) and* $Mann$ *(1.10) iterative schemes gives:*
Let $J = lim_{n\to\infty} \frac{\|x_{n+1}(M-multiIS(C))-p\|}{\|x_{n+1}(Mann)-p\|}$*, then*

$$J \quad = \quad lim_{n\to\infty} \frac{\delta^n[(1-(1-\delta)\delta^{k-2})\beta^1\beta^2\beta^3\ldots\beta^{k-2}\beta^{k-1}]^n\|x_1(Mmulti\ IS)(C)-p\|}{[(1-(1-\delta))\alpha]^n\|x_1(Mann)-p\|} = 0.$$

Thus, the modified multistep iterative scheme (C) *(2.19) converges to the fixed point* p *faster than Mann iterative scheme (1.10) to* p*.*

Case 8. *Comparison of* $Mmultistep\ IS$ (C) *(2.19) and* $Picard$ *(1.9) iterative schemes gives:*
Let $J = lim_{n\to\infty} \frac{\|x_{n+1}(M-multiIS(C))-p\|}{\|x_{n+1}(Picard)-p\|}$*, then*

$$J \quad = \quad lim_{n\to\infty} \frac{\delta^n[(1-(1-\delta)\delta^{k-2})\beta^1\beta^2\beta^3\ldots\beta^{k-2}\beta^{k-1}]^n\|x_1(Mmulti\ IS)(C)-p\|}{[\delta^n]\|x_1(Picard)-p\|} = 0.$$

Thus, the modified multistep iterative scheme (C) *(2.19) converges to the fixed point* p *faster than Picard iterative scheme (1.9) to* p*.*

Case 9. *Comparison of Mmultistep IS (A) (2.17) and T-IS (1.16) gives:*
Let $J = lim_{n\to\infty} \frac{\|x_{n+1}(MmultiIS(A))-p\|}{\|x_{n+1}(T-IS)-p\|}$*, then*

$$J = lim_{n\to\infty} \frac{\delta^n[(1-(1-\delta^{k-1}))\alpha\beta^1\beta^2\beta^3\ldots\beta^{k-2}\beta^{k-1}]^n\|x_1(MmultiIS)-p\|}{\delta^n[(1-(1-\delta^2))\alpha\beta^1\beta^2]^n\|x_1(TIS)-p\|} = \infty.$$

Thus, Thakur et al. iterative scheme (1.16) converges to the fixed point p *faster than modified multistep iterative scheme* (A) *(2.17) to* p*.*

5. Numerical examples

Example 5.2. *Let* E *be the set of real numbers and* $C = [0, 50]$*. Let* $T : C \to C$ *be a mapping defined by* $Tx = \sqrt{x^2 - 8x + 40}\ \forall x \in C$*. Choose* $\alpha_n = 0.85$*,* $\beta_n^1 = 0.65$*,* $\beta_n^2 = 0.45$*,* $\beta_n^3 = 0.25$*,* $\beta_n^4 = 0.05$*. Let the initial value be* $x_1 = 40 \in C$*, and the fixed point* $p = 5.0 \in C$*.*

We now present the convergence speed of the various iterative schemes under this study in Figure 1. It is already known in [32] that iterative scheme (1.16) is faster than (1.9), (1.10), (1.12) to (1.16). We only need to compare iterative schemes (2.17) to (2.19), (1.11) with (1.16). The fixed point of T is $p = 5$ and all the iterative schemes converge to p.

The approximate values of modified multistep iterative scheme (Mmulti IS) (A) (2.17), modified multistep iterative scheme (Mmulti IS) (B) (2.18), modified multistep iterative scheme (Mmulti IS) (C) (2.19), Picard-Mann iterative scheme PMannIS) (1.11) and Thakur et al. iterative scheme (TIS) (1.16) to their fixed points are shown in Figure 1.

TABLE 1. The Convergence Speed of the Iterative Schemes on Example 5.2

No.	Picard	Mann	Picard-Mann	Ishikawa	Agarwal	Noor	Abass-Nazir	Thakur	Mmulti-SA	Mmulti-SB	Mmulti-SC
1	40.0000	40.0000	40.0000	40.0000	40.0000	40.0000	40.0000	40.0000	40.0000	40.0000	40.0000
2	36.3318	36.8820	33.2450	34.8752	34.3249	33.9816	34.2400	32.9459	33.1994	32.0321	31.2733
3	32.7008	33.7905	26.6456	29.8335	28.7529	28.0883	28.5874	26.0697	26.5642	24.3100	22.8544
4	29.1160	30.7306	20.2911	24.9067	23.3290	22.3812	23.0905	19.4826	20.1921	17.0264	15.0287
5	25.5893	27.7091	14.3649	20.1467	18.1322	16.9736	17.8351	13.4242	14.2849	10.6623	8.6021
6	22.1381	24.7348	9.2952	15.6449	13.3147	12.0962	12.9887	8.4746	9.3072	6.4343	5.4478
7	18.7881	21.8200	6.0360	11.5741	9.1939	8.2289	8.9032	5.7280	6.1764	5.1555	5.0238
8	15.5784	18.9820	5.0992	8.2639	6.3717	6.0182	6.2124	5.0765	5.1712	5.0115	5.0011
9	12.5722	16.2455	5.0066	6.1737	5.2434	5.2517	5.2065	5.0065	5.0187	5.0008	5.0001
10	9.8733	13.6476	5.0004	5.3185	5.0298	5.0576	5.0253	5.0005	5.0019	5.0001	5.0000
11	7.6483	11.2443	5.0000	5.0769	5.0034	5.0130	5.0029	5.0000	5.0002	5.0000	5.0000
12	6.1082	9.1201	5.0000	5.0180	5.0004	5.0029	5.0003	5.0000	5.0000	5.0000	5.0000
13	5.3333	7.3914	5.0000	5.0042	5.0000	5.0006	5.0000	5.0000	5.0000	5.0000	5.0000
14	5.0772	6.1733	5.0000	5.0010	5.0000	5.0001	5.0000	5.0000	5.0000	5.0000	5.0000
15	5.0160	5.4815	5.0000	5.0002	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000
16	5.0032	5.1726	5.0000	5.0001	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000
17	5.0006	5.0576	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000
18	5.0001	5.0187	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000
19	5.0000	5.0060	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000
20	5.0000	5.0019	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000
21	5.0000	5.0006	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000
22	5.0000	5.0002	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000
23	5.0000	5.0001	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000
24	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000
25	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000
26	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000
27	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000

Example 5.3. *Let E be the set of real numbers and $C = [0, 50]$. Let $T : C \to C$ be a mapping defined by $Tx = \sqrt{x^2 - 8x + 40}$ $\forall x \in C$. Choose $\alpha_n = \beta_n^1 = \beta_n^2 = \beta_n^3 = \beta_n^4 = 0.5$. Let the initial value be $x_1 = 40 \in C$ and the fixed point $p = 5.0 \in C = [0, 50]$.*

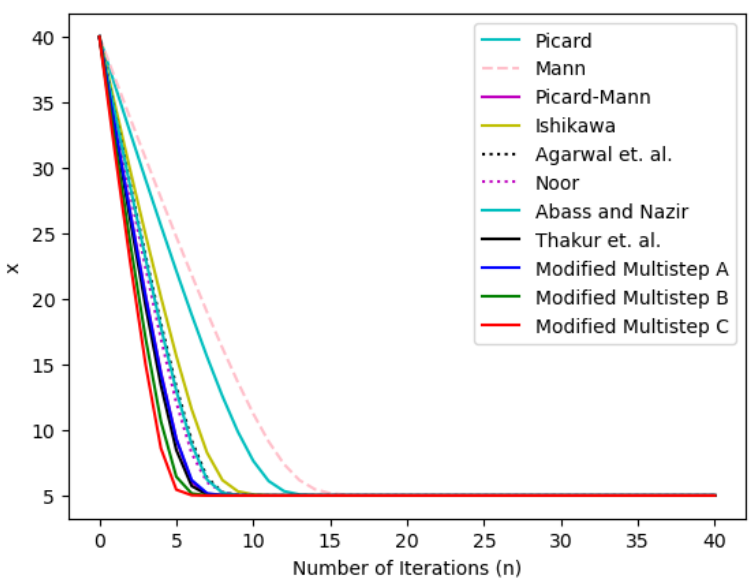

FIGURE 1. Convergence Speed of Fixed-Point Iterative Schemes on Example 5.2

We now present the convergence speed of the various iterative schemes under this study in Figure 2. The fixed point of T is $p = 5$ and all the iterative schemes converge to p.

TABLE 2. The Convergence Speed of the Iterative Schemes on Example 5.3

No.	Picard	Mann	Picard-Mann	Ishikawa	Agarwal	Noor	Abass-Nazir	Thakur	Mmulti-SA	Mmulti-SB	Mmulti-SC
1	40.0000	40.0000	40.0000	40.0000	40.0000	40.0000	40.0000	40.0000	40.0000	40.0000	40.0000
2	36.3318	38.1659	34.5153	37.2577	35.4236	36.8083	33.1593	34.5207	34.6420	32.7335	29.1352
3	32.7008	36.3406	29.1257	34.5371	30.9095	33.6480	26.4817	29.1385	29.3792	25.6736	18.8447
4	29.1160	34.5251	23.8704	31.8420	26.4778	30.5259	20.0639	23.8940	24.2505	18.9625	10.0100
5	25.5893	32.7204	18.8168	29.1773	22.1589	27.4507	14.1073	18.8574	19.3222	12.9216	5.4701
6	22.1381	30.9278	14.0903	26.5488	18.0021	24.4344	9.0868	14.1593	14.7143	8.2836	5.0136
7	18.7881	29.1488	9.9440	23.9647	14.0920	21.4936	5.9870	10.0601	10.6566	5.8526	5.0004
8	15.5784	27.3852	6.8749	21.4357	10.5818	18.6520	5.1076	7.0387	7.5503	5.1324	5.0000
9	12.5722	25.6390	5.4025	18.9761	7.7455	15.9440	5.0085	5.5150	5.7833	5.0158	5.0000
10	9.8733	23.9128	5.0557	16.6061	5.9443	13.4198	5.0007	5.0856	5.1607	5.0018	5.0000
11	7.6483	22.2097	5.0068	14.3537	5.2133	11.1507	5.0000	5.0121	5.0266	5.0002	5.0000
12	6.1082	20.5334	5.0008	12.2582	5.0373	9.2258	5.0000	5.0016	5.0042	5.0000	5.0000
13	5.3333	18.8887	5.0001	10.3725	5.0061	7.7244	5.0000	5.0002	5.0007	5.0000	5.0000
14	5.0772	17.2813	5.0000	8.7603	5.0010	6.6636	5.0000	5.0000	5.0001	5.0000	5.0000
15	5.0160	15.7187	5.0000	7.4788	5.0002	5.9777	5.0000	5.0000	5.0000	5.0000	5.0000
16	5.0032	14.2101	5.0000	6.5468	5.0000	5.5611	5.0000	5.0000	5.0000	5.0000	5.0000
17	5.0006	12.7673	5.0000	5.9257	5.0000	5.3176	5.0000	5.0000	5.0000	5.0000	5.0000
18	5.0001	11.4053	5.0000	5.5388	5.0000	5.1784	5.0000	5.0000	5.0000	5.0000	5.0000
19	5.0000	10.1422	5.0000	5.3085	5.0000	5.0997	5.0000	5.0000	5.0000	5.0000	5.0000
20	5.0000	8.9994	5.0000	5.1749	5.0000	5.0556	5.0000	5.0000	5.0000	5.0000	5.0000
21	5.0000	7.9995	5.0000	5.0986	5.0000	5.0310	5.0000	5.0000	5.0000	5.0000	5.0000
22	5.0000	7.1619	5.0000	5.0555	5.0000	5.0172	5.0000	5.0000	5.0000	5.0000	5.0000
23	5.0000	6.4963	5.0000	5.0311	5.0000	5.0096	5.0000	5.0000	5.0000	5.0000	5.0000
24	5.0000	5.9973	5.0000	5.0175	5.0000	5.0053	5.0000	5.0000	5.0000	5.0000	5.0000
25	5.0000	5.6439	5.0000	5.0098	5.0000	5.0030	5.0000	5.0000	5.0000	5.0000	5.0000
26	5.0000	5.4057	5.0000	5.0055	5.0000	5.0016	5.0000	5.0000	5.0000	5.0000	5.0000
27	5.0000	5.2512	5.0000	5.0031	5.0000	5.0009	5.0000	5.0000	5.0000	5.0000	5.0000
28	5.0000	5.1537	5.0000	5.0017	5.0000	5.0005	5.0000	5.0000	5.0000	5.0000	5.0000
29	5.0000	5.0933	5.0000	5.0010	5.0000	5.0003	5.0000	5.0000	5.0000	5.0000	5.0000
30	5.0000	5.0564	5.0000	5.0005	5.0000	5.0002	5.0000	5.0000	5.0000	5.0000	5.0000
31	5.0000	5.0340	5.0000	5.0003	5.0000	5.0001	5.0000	5.0000	5.0000	5.0000	5.0000
32	5.0000	5.0205	5.0000	5.0002	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000
33	5.0000	5.0123	5.0000	5.0001	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000
34	5.0000	5.0074	5.0000	5.0001	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000
35	5.0000	5.0044	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000
36	5.0000	5.0027	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000
37	5.0000	5.0016	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000
38	5.0000	5.0010	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000
39	5.0000	5.0006	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000
40	5.0000	5.0003	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000

The approximate values of Picard (1.9), Mann (1.10), Picard- Mann (1.11), Ishikawa (1.12), Agawalel at. (1.13), Noor (1.14), Abbas and Nazir (1.15), Thakur et al. (1.16), modified multistep $(MmultiIS)$ (A) (2.17), modified multistep $(MmultiIS)$ (B) (2.18) and modified multistep$(MmultiIS)$ (C) (2.19) iterative schemes to their fixed points are shown in Figure 2.

6. CONCLUSION

This research is novel. The convergence speed of several iterative schemes were proven to the fixed point analytically and numerically. The numerical results were represented graphically in Figures 1 and 2.

The graph illustrated that the multistep iterative scheme C converged to the fixed point $p = 5.0$ in lesser number of iterations than some of the existing iterative schemes in the literature including Thakur et al. [31]. With the help of well constructed theorems the modified iterative schemes A, B and C were applied to constrained minimization and split feasibility problems for the class of nonexpansive mappings in Hilbert spaces. The

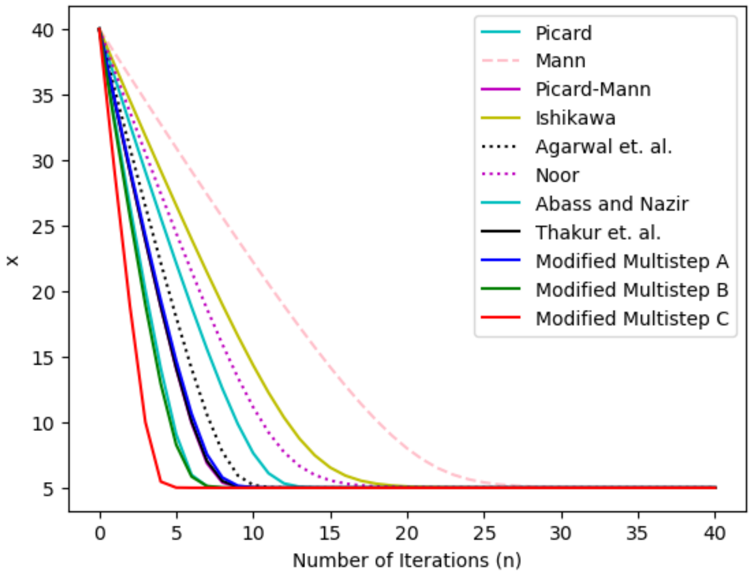

FIGURE 2. Convergence Speed of Fixed-Point Iterative Schemes on Example 5.3

various iterative schemes and classes of mappings considered in this study have good potentials for further research.

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