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Reciprocal distance spectrum of Indu-Bala product of graphs

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ABSTRACT. Let G be a simple connected graph. The reciprocal distance spectrum of G is the multiset of reciprocal distance eigenvalues of G. The Indu-Bala product $G\blacktriangledown H$ of two graphs G and H is obtained from two disjoint copies of the join $G \vee H$ of G and H by joining the corresponding vertices in the two copies of H. In this paper we obtain the reciprocal distance spectrum of $G\blacktriangledown H$ in terms of adjacency eigenvalues of G and H. We also construct infinitely many non-isomorphic pairs of reciprocal distance equienergetic graphs and reciprocal distance cospectral graphs of diameter 3.

1. INTRODUCTION

In this paper we consider undirected, finite and simple graphs. Let $G = (V(G), E(G))$ be a graph of order n with $V(G) = \{z_1, z_2, \ldots, z_n\}$. Let d_{jk} denote the distance between z_j and z_k. The largest distance between any pair of distinct vertices in G is the diameter of G. We refer [13] for graph theoretic terminology.

There are several matrices associated with a graph and many of them are found to be applicable in theoretical physics, quantum mechanics, molecular chemistry, data communication systems etc[6, 17]. Exploring the characteristic polynomials and eigenvalues associated with these matrices has been a focal point of research within spectral graph theory. A significant amount of structural information about the graph can be inferred from the spectrum of these matrices. The simplest way by which a graph G can be represented using a matrix is its adjacency matrix $A(G)$. The adjacency characteristic polynomial in μ is defined by $\Phi\big(A(G);\mu\big) = det\big(\mu I - A(G)\big)$. The eigenvalues of $A(G)$ are the adjacency eigenvalues (or, simply eigenvalues)[6] of G denoted by $\mu_1(G) \geq \mu_2(G) \geq \ldots \geq \mu_n(G)$ and their collection is the spectrum $Spec(G)$ of G. If $\mu_{n_1}(G), \ldots, \mu_{n_t}(G)$ are the distinct eigenvalues of G with multiplicities $m_1, \ldots, m_t$ then we write $Spec(G) = \{\mu_{n_1}(G)^{m_1}, \ldots, \mu_{n_t}(G)^{m_t}\}$. $\mu_1(G)$ is of multiplicity 1, when G is connected. If G is r-regular then $\mu_1(G) = r$. G is cospectral with a graph G' if $Spec(G) = Spec(G')$.

The energy $E(G)$, of G is defined as the sum of the absolute values of the adjacency eigenvalues[11]. i.e.,

$$E(G) = \sum_{j=1}^{n} |\mu_j(G)|\,. \tag{1.1}$$

The concept of the energy of a graph emerged from the field of theoretical chemistry with roots in the study of molecular graphs and their associated energy, which is calculated using the Hückel molecular orbital theory. For more details on energy refer [1, 12, 18].

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The reciprocal distance(RD) matrix was introduced by Ivancius et.al[16], and found applicable in the study of molecular graphs in quantitative structure - property relationships ($QSPR$) and quantitative structure - activity relationships ($QSAR$) models. The reciprocal distance matrix of a connected graph G, also known as Harary matrix[24], is a square matrix of order n defined as $RD(G) = [h_{jk}]$, where

$$h_{jk} = \begin{cases} \frac{1}{d_{jk}} & j \neq k \\ 0 & \text{otherwise} \end{cases}$$

Let $\Phi\big(RD(G);\zeta\big)$ denote the reciprocal distance characteristic polynomial in ζ of G. Being a real symmetric matrix the eigenvalues of $RD(G)$ are real numbers and are called the reciprocal distance(RD) eigenvalues(or, Harary eigenvalues) of G; labeled as $\zeta_1(G) \geq \zeta_2(G) \geq \ldots \geq \zeta_n(G)$ constitute the reciprocal distance spectrum(or, Harary spectrum) of G denoted by $Spec_{RD}(G)$. If $\zeta_{n_1}(G), \ldots, \zeta_{n_t}(G)$ are the distinct RD-eigenvalues of G with multiplicities $m_1, \ldots, m_t$ then we write $Spec_{RD}(G) = \{\zeta_{n_1}(G)^{m_1}, \ldots, \zeta_{n_t}(G)^{m_t}\}$. The largest RD-eigenvalue $\zeta_1(G)$ is the reciprocal distance spectral radius of the graph G, extensively studied in [5, 7, 31]. Two non-isomorphic connected graphs are reciprocal distance cospectral[26] if their RD-eigenvalues are the same.
The reciprocal distance index(or, Harary index) of G is defined as half the sum of the entries of $RD(G)$. It is a well-known topological index with various physico-chemical properties. Relations between reciprocal distance index and several graph properties are obtained in[9]. The reciprocal distance energy(or, Harary energy) of G, denoted by $RDE(G)$, is defined analogous to equation (1.1). i.e.,

$$RDE(G) = \sum_{j=1}^{n} |\zeta_j(G)|. \tag{1.2}$$

Results on reciprocal distance energy are found in [5, 10, 25, 26]. Two non-isomorphic connected graphs are reciprocal distance equienergetic[25] if they have the same reciprocal distance energy.
Graph operations hold significance within spectral graph theory as they are applied in constructing several graph classes with special structures and properties. In specific scenarios, the spectrum of a comparatively larger graph can be described in terms of the spectra of smaller graphs (graphs involved in the operation) through graph operations such as disjoint union, graph join, edge deletion/insertion, and graph complement etc. This allows for the representation of a complex network using small, easily recognizable graphs whose spectra can be computed easily. For more details see the survey by Barik et al.[3].
The line graph $L(G)$ of G is the graph whose vertices are the edges of G and two vertices of $L(G)$ are adjacent if the corresponding edges in G have a common vertex[13]. If G is r-regular and has order n then $L(G)$ is $2r-2$ regular with order $\frac{nr}{2}$. For a positive integer $m \geq 2$, let $L^m(G)$ denote the m^{th} iterated line graph, defined by $L^m(G) = L(L^{m-1}(G))$, where $L^1(G) = L(G)$[13]. Clearly $L^m(G)$ is regular for all m when G is regular. For $m \geq 1$ let $L^m(G)$ be r_m-regular on n_m vertices. Then $n_m = \dfrac{n_{m-1} \cdot r_{m-1}}{2}$ and $r_m = 2r_{m-1} - 2$, with $n_0 = n$ and $r_0 = r$.
It can be deduced that

$$n_m = \frac{n}{2^m} \prod_{i=0}^{m-1} \left(2^i r - 2^{i+1} + 2\right) \tag{1.3}$$

and

$$r_m = 2^m r - 2^{m+1} + 2. \tag{1.4}$$

The Indu-Bala product $G\blacktriangledown H$, of two graphs G and H, is obtained from two disjoint copies of the join $G \vee H$ of G and H by joining the corresponding vertices in the two copies of H[15]. This graph operation has garnered significant attention from researchers in different areas of graph theory. Various Spectral studies of Indu-Bala product can be seen in [2, 15, 21, 22]. Researches on degree based and distance based topological indices of Indu-Bala product are reported in [14, 20, 27, 29].

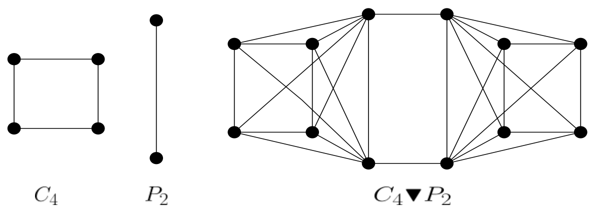

FIGURE 1. Indu-Bala product of C_4 and P_2

In this paper, we obtain the RD-characteristic polynomial of $G\blacktriangledown H$, when G is regular and H is an arbitrary graph. We completely describe the RD-spectrum of $G\blacktriangledown H$ in terms of adjacency eigenvalues of G and H, when both G and H are regular. As applications we construct infinitely many pairs of non-isomorphic RD-equienergetic graphs and RD-cospectral graphs of diameter 3.
Throughout this paper J denote the the all-one matrix of appropriate order.

2. PRELIMINARIES

We recall the following definitions and results.

Proposition 2.1. [30] *Let P_1, P_2, P_3 and P_4 be matrices with P_4 invertible and* $P = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}$. *Then $det(P) = det(P_4) \cdot det\left(P_1 - P_2 P_4^{-1} P_3\right)$, where $P_1 - P_2P_4^{-1}P_3$ is the Schur complement of P_4 in M.*

Proposition 2.2. [8] *Let P_1 and P_2 be square matrices of the same order and* $P = \begin{bmatrix} P_1 & P_2 \\ P_2 & P_1 \end{bmatrix}$. *Then the eigenvalues of P are those of $P_1 + P_2$ together with those of $P_1 - P_2$.*

Definition 2.1. [4, 19] *The coronal $\Gamma_G(\mu)$ of a graph G on n vertices is the sum of the elements of $\big(\mu I - A(G)\big)^{-1}$. i.e., $\Gamma_G(\mu) = \mathbf{1}_n^T \big(\mu I - A(G)\big)^{-1} \mathbf{1}_n$, where $\mathbf{1}_n$ denote the all-one column vector of size n.*

The above definition can be generalized for any graph matrix M.

Definition 2.2. [4] *Let M be an $n \times n$ matrix associated with a graph G on n vertices. The M-coronal $\Gamma_M(\mu)$ is defined as $\Gamma_M(\mu) = \mathbf{1}_n^T \big(\mu I - M\big)^{-1} \mathbf{1}_n$.*

Proposition 2.3. [4] *Let M be an $n \times n$ graph matrix associated with G. If each row sum of M is s, then $\Gamma_M(\mu) = \dfrac{n}{\mu - s}$.*

Theorem 2.1. [6] *Let G be an r-regular graph on n vertices with $Spec(G) = \{r, \mu_2(G), \ldots, \mu_n(G)\}$. Then $Spec(L(G))$ consists of $\mu_j(G) + r - 2, j = 1, \ldots, n$, each with multiplicity 1 and -2 with multiplicity $n\Big(\frac{r}{2} - 1\Big)$.*

Theorem 2.2. [6] *Let G be an r-regular graph on n vertices with $Spec(G) = \{r, \mu_2(G), \ldots, \mu_n(G)\}$. Then the complement $\overline{G}$ has the adjacency spectrum $Spec(\overline{G}) = \{n-r-1, -(\mu_2(G)+1), \ldots, -(\mu_n(G)+1)\}$.*

Theorem 2.3. [28] *Let G and G' be r-regular graphs on n vertices. Then $L^m(G)$ and $L^m(G')$, $m \geq 1$, have the same order and size. Moreover, $L^m(G)$ and $L^m(G')$ are cospectral if and only if G and G' are cospectral.*

Definition 2.3. [23] *The strong double graph $D_2^*(G)$ of a graph G is the graph formed by taking two copies of G say G_1 and G_2 and joining each vertex u_1 in G_1 to the corresponding vertex u_2 in G_2 and also to the neighbours of u_2.*

Theorem 2.4. [23] *Let G be a graph with $Spec(G) = \{\mu_1(G), \ldots, \mu_n(G)\}$. Then $Spec(D_2^*(G))$ consists of $2\mu_1(G) + 1, \ldots, 2\mu_n(G) + 1$, each with multiplicity 1 and -1 with multiplicity n.*

3. The Reciprocal Distance spectrum of $G\blacktriangledown H$

Theorem 3.5. *Let G be an r-regular graph with $Spec(G) = \{r, \mu_2(G), \ldots, \mu_n(G)\}$ and H be a graph of order p. Then $\Phi\big(RD(G\blacktriangledown H); x\big) =$*

$$\Phi\left(\frac{5}{6}J + \frac{2}{3}A(H); x - \frac{1}{6}\right) \cdot \left(x - \frac{1}{2}(r-1) - n\left(\frac{5}{6} + \frac{9}{4}\Gamma_{\frac{5}{6}J + \frac{2}{3}A(H)}(x - 1/6)\right)\right) \cdot$$
$$\Phi\left(\frac{1}{6}J + \frac{1}{3}A(H); x + \frac{7}{6}\right) \cdot \left(x - \frac{1}{2}(r-1) - n\left(\frac{1}{6} + \frac{1}{4}\Gamma_{\frac{1}{6}J + \frac{1}{3}A(H)}(x + 7/6)\right)\right) \cdot$$
$$\prod_{j=2}^{n}\left(x - \frac{1}{2}\Big(\mu_j(G) - 1\Big)\right)^2.$$

Proof. Arranging the vertices in proper way, $RD(G\blacktriangledown H)$ has the form

$$RD(G\blacktriangledown H) = \left[\begin{array}{cc|cc} \frac{1}{2}\big(J - I + A(G)\big) & J_{n\times p} & \frac{1}{3}J_{n\times n} & \frac{1}{2}J_{n\times p} \\ J_{p\times n} & \frac{1}{2}\big(J - I + A(H)\big) & \frac{1}{2}J_{p\times n} & \frac{1}{3}\big(J + 2I + \frac{1}{2}A(H)\big) \\ \hline \frac{1}{3}J_{n\times n} & \frac{1}{2}J_{n\times p} & \frac{1}{2}\big(J - I + A(G)\big) & J_{n\times p} \\ \frac{1}{2}J_{p\times n} & \frac{1}{3}\big(J + 2I + \frac{1}{2}A(H)\big) & J_{p\times n} & \frac{1}{2}\big(J - I + A(H)\big) \end{array}\right],$$

which is a 2×2 block symmetric matrix of the form

$$\begin{bmatrix} P_1 & P_2 \\ P_2 & P_1 \end{bmatrix},$$

where

$$P_1 = \begin{bmatrix} \frac{1}{2}\big(J - I + A(G)\big) & J_{n\times p} \\ J_{p\times n} & \frac{1}{2}\big(J - I + A(H)\big) \end{bmatrix}$$

and

$$P_2 = \begin{bmatrix} \frac{1}{3}J_{n\times n} & \frac{1}{2}J_{n\times p} \\ \frac{1}{2}J_{p\times n} & \frac{1}{3}\big(J + 2I + \frac{1}{2}A(H)\big) \end{bmatrix}.$$

By proposition 2.2, the RD-spectrum of $G\blacktriangledown H$ is the set of eigenvalues of $P_1 + P_2$ together with those of $P_1 - P_2$. By proposition 2.1,

$\Phi(P_1+P_2;x) = det\Big(\big(x-\frac{1}{6}\big)I - \big(\frac{5}{6}J+\frac{2}{3}A(H)\big)\Big)\cdot$
$det\bigg(\big(x+\frac{1}{2}\big)I - \big(\frac{5}{6}J+\frac{1}{2}A(G)\big) - \frac{9}{4}J_{n\times p}\Big(\big(x-\frac{1}{6}\big)I - \big(\frac{5}{6}J+\frac{2}{3}A(H)\big)\Big)^{-1}J_{p\times n}\bigg).$

By definition 2.2,

$$J_{n\times p}\Big(\big(x-\tfrac{1}{6}\big)I - \big(\tfrac{5}{6}J+\tfrac{2}{3}A(H)\big)\Big)^{-1}J_{p\times n} = \Gamma_{\frac{5}{6}J+\frac{2}{3}A(H)}(x-1/6)J_{n\times n}.$$

Therefore,

$$\Phi(P_1+P_2;x) =$$
$$\Phi\left(\frac{5}{6}J+\frac{2}{3}A(H);x-\frac{1}{6}\right)\cdot\left(x-\frac{1}{2}(r-1)-n\left(\frac{5}{6}+\frac{9}{4}\Gamma_{\frac{5}{6}J+\frac{2}{3}A(H)}\left(x-1/6\right)\right)\right)\cdot$$
$$\prod_{j=2}^{n}\left(x-\frac{1}{2}\big(\mu_j(G)-1\big)\right).$$

Proceeding in the same way, we obtain

$$\Phi(P_1-P_2;x) =$$
$$\Phi\left(\frac{1}{6}J+\frac{1}{3}A(H);x+\frac{7}{6}\right)\cdot\left(x-\frac{1}{2}(r-1)-n\left(\frac{1}{6}+\frac{1}{4}\Gamma_{\frac{1}{6}J+\frac{1}{3}A(H)}\left(x+7/6\right)\right)\right)$$
$$\prod_{j=2}^{n}\left(x-\frac{1}{2}\big(\mu_j(G)-1\big)\right).$$

Then the result follows since $\Phi(RD(G\blacktriangledown H);x) = \Phi(P_1+P_2;x)\cdot\Phi(P_1-P_2;x)$. □

Corollary 3.1. *Let G be an r-regular graph with $Spec(G) = \{r,\mu_2(G),\ldots,\mu_n(G)\}$ and H be a k-regular graph with $Spec(H) = \{k,\mu_2(H),\ldots,\mu_p(H)\}$. Then the RD-eigenvalues of $G\blacktriangledown H$ are $\frac{1}{2}\big(\mu_j(G)-1\big), j = 2,3,\ldots,n$, each with multiplicity 2;*
$\frac{1}{3}\left(2\mu_j(H)+\frac{1}{2}\right)$ and $\frac{1}{3}\left(\mu_j(H)-\frac{7}{2}\right)$, $j = 2,3,\ldots,p$; and the four numbers which are the solutions of the equation

$$\left(\left(x+\frac{1}{2}-\frac{5}{6}n-\frac{1}{2}r\right)\left(x-\frac{1}{6}-\frac{5}{6}p-\frac{2}{3}k\right)-\frac{9}{4}np\right)\cdot \tag{3.5}$$
$$\left(\left(x+\frac{1}{2}-\frac{1}{6}n-\frac{1}{2}r\right)\left(x+\frac{7}{6}-\frac{1}{6}p-\frac{1}{3}k\right)-\frac{1}{4}np\right)=0.$$

Proof. By theorem 3.5, we obtain directly that $\frac{1}{2}\big(\mu_j(G)-1\big), j = 2,3,\ldots,n$ are RD-eigenvalues of $G\blacktriangledown H$, each with multiplicity 2.
Again by theorem 3.5, both

$$\Phi\left(\frac{5}{6}J+\frac{2}{3}A(H);x-\frac{1}{6}\right) = \left(x-\frac{1}{6}-\left(\frac{5}{6}p+\frac{2}{3}k\right)\right)\cdot\prod_{j=2}^{p}\left(x-\frac{1}{6}-\frac{2}{3}\mu_j(H)\right) \tag{3.6}$$

and

$$\Phi\left(\frac{1}{6}J+\frac{1}{3}A(H);x+\frac{7}{6}\right)=\left(x+\frac{7}{6}-\left(\frac{1}{6}p+\frac{1}{3}k\right)\right)\cdot\prod_{j=2}^{p}\left(x+\frac{7}{6}-\frac{1}{3}\mu_j(H)\right) \tag{3.7}$$

are factors of the characteristic polynomial $\Phi\big(RD(G\blacktriangledown H);x\big)$. Hence we get $\frac{1}{3}\left(2\mu_j(H)+\frac{1}{2}\right)$ and $\frac{1}{3}\left(\mu_j(H)-\frac{7}{2}\right), j=2,3,\ldots,p$ are also RD-eigenvalues of $G\blacktriangledown H$.

Now, each row sum of the matrices $\frac{5}{6}J+\frac{2}{3}A(H)$ and $\frac{1}{6}J+\frac{1}{3}A(H)$ are respectively $\frac{5}{6}p+\frac{2}{3}k$ and $\frac{1}{6}p+\frac{1}{3}k$. Then by proposition 2.3,

$$\Gamma_{\frac{5}{6}J+\frac{2}{3}A(H)}\left(x-1/6\right)=\frac{p}{x-\frac{1}{6}-\left(\frac{5}{6}p+\frac{2}{3}k\right)} \tag{3.8}$$

and

$$\Gamma_{\frac{1}{6}J+\frac{1}{3}A(H)}\left(x+7/6\right)=\frac{p}{x+\frac{7}{6}-\left(\frac{1}{6}p+\frac{1}{3}k\right)} \tag{3.9}$$

Substituting (3.8) and (3.9) in the expression for $\Phi\big(RD(G\blacktriangledown H);x\big)$ in theorem 3.5 and using (3.6) and (3.7) we obtain the remaining four RD-eigenvalues as the solutions of the biquadratic equation (3.5). □

Example 3.1. *We know that $Spec(C_4)=\{-2,0^2,2\}$ and $Spec(P_2)=\{-1,1\}$. Therefore RD-spectrum of $C_4\blacktriangledown P_2$, shown in figure 1 is*
$Spec_{RD}(C_4\blacktriangledown P_2)=\left\{\left(\frac{-3}{2}\right)^3,\left(\frac{-1}{2}\right)^5,\frac{1}{3}+\frac{\sqrt{97}}{6},\frac{1}{3}-\frac{\sqrt{97}}{6},\frac{19}{6}+\frac{\sqrt{166}}{3},\frac{19}{6}-\frac{\sqrt{166}}{3}\right\}$

4. RD-EQUIENERGETIC GRAPHS

We construct infinitely many pairs of RD-non-cospectral and RD-equienergetic graphs of diameter 3 using corollary 3.1.

In the subsequent theorems in this section, we use the notation

$$S_H=\frac{1}{3}\sum_{j=2}^{p}\left|2\mu_j(H)+\frac{1}{2}\right|+\frac{1}{3}\sum_{j=2}^{p}\left|\mu_j(H)-\frac{7}{2}\right|.$$

Theorem 4.6. *Let G and G' be r-regular non-cospectral graphs on n vertices and H be a k-regular graph on p vertices. Suppose that $\mu_n(G),\mu_n(G')\geq 3-r$. Then $L(G)\blacktriangledown H$ and $L(G')\blacktriangledown H$ form RD-non-cospectral and RD-equienergetic graphs.*

Proof. By corollary 3.1 and theorem 2.1, we obtain the RD-eigenvalues of $L(G)\blacktriangledown H$ as

$$\begin{aligned}
&\frac{1}{2}(\mu_j(G)+r-3),\quad \text{with multiplicity 2},\quad j=2,3,\ldots,n;\\
&-\frac{3}{2},\quad \text{with multiplicity } n(r-2);\\
&\frac{1}{3}\left(2\mu_j(H)+\frac{1}{2}\right),\quad j=2,3,\ldots,p;\\
&\frac{1}{3}\left(\mu_j(H)-\frac{7}{2}\right),\quad j=2,3,\ldots,p;
\end{aligned}$$

and four more numbers that are the solutions of the equation

$$\left(\left(x-\frac{5}{12}nr-r+\frac{3}{2}\right)\left(x-\frac{5}{6}p-\frac{2}{3}k-\frac{1}{6}\right)-\frac{9}{8}nrp\right)\cdot \left(\left(x-\frac{1}{12}nr-r+\frac{3}{2}\right)\left(x-\frac{1}{6}p-\frac{1}{3}k+\frac{7}{6}\right)-\frac{1}{8}nrp\right)=0, \tag{4.10}$$

In the same way, we obtain the reciprocal distance eigenvalues of $L(G')\blacktriangledown H$ and we can see that $L(G)\blacktriangledown H$ and $L(G')\blacktriangledown H$ are RD-non-cospectral. Since the solutions of (4.10) depend only on the parameters n, r, p and k, let us denote the sum of the absolute values of the solutions as $f(n,r,p,k)$. Then

$$RDE\big(L(G)\blacktriangledown H\big)=\sum_{j=2}^{n}\left|\mu_j(G)+r-3\right|+3n\Big(\frac{r}{2}-1\Big)+S_H+f(n,r,p,k). \tag{4.11}$$

Since $\mu_n(G)\geq 3-r$, we have $\mu_j(G)+r-3\geq 0$ for $j=2,3,\ldots,n$. Then from equation (4.11),

$$RDE\big(L(G)\blacktriangledown H\big)=\frac{5}{2}nr-(6n+2r-3)+S_H+f(n,r,p,k).$$

By the same set of arguments, we can compute the reciprocal distance energy of $L(G')\blacktriangledown H$. We can observe that $RDE\big(L(G)\blacktriangledown H\big)=RDE\big(L(G')\blacktriangledown H\big)$. □

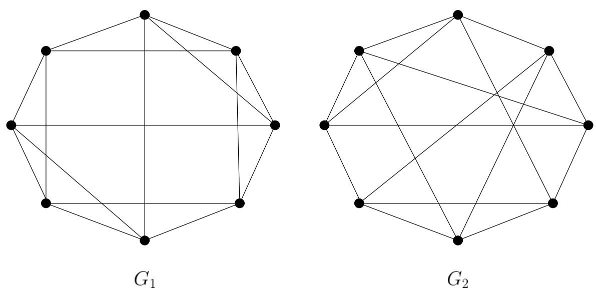

FIGURE 2. Non-isomorphic 4-regular graphs on 8 vertices

Example 4.2. *Consider the graphs G_1 and G_2 in figure 2. Let $G=L(G_1), G'=L(G_2)$ and $H=C_4$. Then $Spec(G)=\{(-2)^8,-\sqrt{3}+1,0,2-\sqrt{2},2^2,\sqrt{3}+1,2+\sqrt{2},6\}$ and $Spec(G')=\{(-2)^8,1-\sqrt{5},0,2^4,1+\sqrt{5},6\}$. Note that $\mu_{16}(G),\mu_{16}(G')>-3$. Then $Spec_{RD}(L(G)\blacktriangledown H)=\Big\{\big(\frac{-3}{2}\big)^{64},\big(\frac{1}{2}\big)^{16},\Big(2-\frac{\sqrt{3}}{2}\Big)^2,\big(\frac{3}{2}\big)^2,\Big(\frac{5-\sqrt{2}}{2}\Big)^2,\big(\frac{5}{2}\big)^4,\Big(2+\frac{\sqrt{3}}{2}\Big)^2,\Big(\frac{5+\sqrt{2}}{2}\Big)^2,\big(\frac{1}{6}\big)^2,\big(-\frac{7}{6}\big)^3,-\frac{11}{6},\frac{74}{3}+\frac{\sqrt{29713}}{6},\frac{74}{3}-\frac{\sqrt{29713}}{6},\frac{19}{3}+\frac{\sqrt{3097}}{6},\frac{19}{3}-\frac{\sqrt{3097}}{6}\Big\}$ and $Spec_{RD}(L(G')\blacktriangledown H)=\Big\{\big(\frac{-3}{2}\big)^{64},\big(\frac{1}{2}\big)^{16},\Big(2-\frac{\sqrt{5}}{2}\Big)^2,\big(\frac{3}{2}\big)^2,\big(\frac{5}{2}\big)^8,\Big(2+\frac{\sqrt{5}}{2}\Big)^2,\big(\frac{1}{6}\big)^2,\big(-\frac{7}{6}\big)^3,-\frac{11}{6},\frac{74}{3}+\frac{\sqrt{29713}}{6},\frac{74}{3}-\frac{\sqrt{29713}}{6},\frac{19}{3}+\frac{\sqrt{3097}}{6},\frac{19}{3}-\frac{\sqrt{3097}}{6}\Big\}$. We get $RDE\big(L(G)\blacktriangledown H\big)=135+\frac{1}{3}\big(17+\sqrt{29713}+\sqrt{3097}\big)=RDE\big(L(G')\blacktriangledown H\big)$.*

Theorem 4.7. *Let G and G' be r-regular non-cospectral graphs on n vertices. Let H be a k-regular graph of order p. Then*
(a) $L^2(G)\blacktriangledown H$ and $L^2(G')\blacktriangledown H$ form RD-non-cospectral and RD-equienergetic graphs if $r\geq 4$.

(b) $L^m(G)\blacktriangledown H$ *and* $L^m(G')\blacktriangledown H$ *form RD-non-cospectral and RD-equienergetic graphs for* $m \geq 3$ *if* $r \geq 3$.

Proof. The eigenvalues of $L^m(G), m \geq 2$, obtained by repeatedly applying theorem 2.1, m times are as follows.

$$\left.\begin{aligned}
&\mu_j(G) + \big(2^m - 1\big)\big(r - 2\big), \quad j = 1, 2, \ldots, n,\\
&\big(2^m - 2\big)r - 2\big(2^m - 1\big), \quad \text{with multiplicity } n\Big(\frac{r}{2} - 1\Big),\\
&\big(2^m - 2^j\big)r - 2\big(2^m - 2^j + 1\big), \quad \text{with multiplicity } M_j, \quad j = 2, 3, \ldots, m,\\
&\text{where} \quad M_j = \frac{n(r-2)}{2}\prod_{l=0}^{j-2}(2^l r - 2^{l+1} + 2).
\end{aligned}\right\} \tag{4.12}$$

By corollary 3.1, the RD-eigenvalues of $L^m(G)\blacktriangledown H, m \geq 2$, are the following.

$$\begin{aligned}
&\frac{1}{2}\Big(\mu_j(G) + (2^m - 1)(r - 2) - 1\Big), \quad j = 2, 3, \ldots, n, \quad \text{with multiplicity } 2;\\
&\frac{1}{2}\Big((2^m - 2)(r - 2) - 3\Big), \quad \text{with multiplicity } n(r-2);\\
&\frac{1}{2}\Big((2^m - 2^j)(r - 2) - 3\Big), \quad \text{with multiplicity } 2M_j, \quad j = 2, 3, \ldots, m;\\
&\frac{1}{3}\left(2\mu_j(H) + \frac{1}{2}\right), \quad j = 2, 3, \ldots, p;\\
&\frac{1}{3}\left(\mu_j(H) - \frac{7}{2}\right), \quad j = 2, 3, \ldots, p;
\end{aligned}$$

and four more numbers satisfying the equation, obtained by replacing n and r by n_m and r_m respectively, given by equations (1.3) and (1.4), in equation (3.5); and denote its solutions by $x_j, j = 1, 2, 3, 4$. Let $f_1(n, r, p, k) = \sum_{j=1}^{4} |x_j|$.
Since $\mu_j(G) \geq -r$ for $j = 2, 3, \ldots, n$, we have

$$\mu_j(G) + (2^m - 1)(r - 2) - 1 \geq (2^m - 2)(r - 2) - 3.$$

(a) For $m = 2, \mu_j(G) + (2^m - 1)(r - 2) - 1 \geq 2r - 7 > 0$ since $r \geq 4$. Therefore

$$\begin{aligned}
RDE\Big(L^2(G)\blacktriangledown H\Big) &= \sum_{j=2}^{n}\Big(\mu_j(G) + 3r - 7\Big) + n\Big(\frac{r}{2} - 1\Big)(2r - 7) +\\
&\quad \frac{3}{2}nr(r-2) + S_H + f_1(n, r, p, k)\\
&= \frac{nr}{2}(5r - 11) - 4r + 7 + S_H + f_1(n, r, p, k).
\end{aligned}$$

(b) For $m \geq 3$ and $r \geq 3$, $(2^m - 2)(r - 2) - 3 > 0$.
Also for $2 < j < m$, $(2^m - 2^j)(r-2) - 3 > 0$ and the case $j = m$ yields -3, independent

of r. Thus

$$\begin{aligned} RDE\Big(L^m(G)\blacktriangledown H\Big) =& \sum_{j=2}^{n}\Big(\mu_j(G)+(2^m-1)(r-2)-1\Big)+ \\ & n\Big(\frac{r}{2}-1\Big)\Big((2^m-2)(r-2)-3\Big)+ \\ & \sum_{j=2}^{m-1} M_j\Big((2^m-2^j)(r-2)-3\Big)+3M_m+ \\ & S_H+f_1(n,r,p,k) \\ =&(2^{m-1}-1)nr^2+\Big(\frac{n}{2}(3-2^{m+1})-2^m\Big)r+ \\ & 2^{m+1}-1+\sum_{j=2}^{m-1} M_j\Big((2^m-2^j)(r-2)-3\Big)+ \\ & 3M_m+S_H+f_1(n,r,p,k). \end{aligned}$$

By employing the same arguments we can compute the reciprocal distance spectrum of $L^m(G')\blacktriangledown H$, $m \geq 2$ and its energy. Evidently the graphs are RD-non-cospectral. In both cases $RDE\Big(L^m(G)\blacktriangledown H\Big) = RDE\Big(L^m(G')\blacktriangledown H\Big)$. □

Example 4.3. *Consider the* 4*-regular non-cospectral graphs* G_1 *and* G_2 *in figure 2 and let* $H = C_4$*. From example 4.2,* $RDE\big(L^2(G_1)\blacktriangledown H\big) = 135+\frac{1}{3}\big(17+\sqrt{29713}+\sqrt{3097}\big) = RDE\big(L^2(G_2)\blacktriangledown H\big)$.

Example 4.4. *Let* G *be the Petersen graph and* $G' = C_5\square P_2$*, where* $\square$ *denotes the Cartesian product of graphs, and* $H = P_2$*. Then* $Spec_{RD}(L^2(G)\blacktriangledown H) = \Big\{\left(\frac{-3}{2}\right)^{31}, \left(\frac{-1}{2}\right)^{11}, 0^8, \left(\frac{3}{2}\right)^{10}, 15+\frac{\sqrt{1165}}{2}, 15-\frac{\sqrt{1165}}{2}, \frac{7}{2}+\sqrt{31}, \frac{7}{2}-\sqrt{31}\Big\}$, $Spec_{RD}(L^2(G')\blacktriangledown H) = \Big\{\left(\frac{-3}{2}\right)^{31}, \left(\frac{-1}{2}\right)^{11}, \left(\frac{1-\sqrt{5}}{4}\right)^4, \left(\frac{5-\sqrt{5}}{4}\right)^4, \left(\frac{1+\sqrt{5}}{4}\right)^4, \left(\frac{3}{2}\right)^2, \left(\frac{5+\sqrt{5}}{4}\right)^4, 15+\frac{\sqrt{1165}}{2}, 15-\frac{\sqrt{1165}}{2}, \frac{7}{2}+\sqrt{31}, \frac{7}{2}-\sqrt{31}\Big\}$, $Spec_{RD}(L^3(G)\blacktriangledown H) = \Big\{\left(\frac{-3}{2}\right)^{121}, \left(\frac{1}{2}\right)^{30}, \left(\frac{3}{2}\right)^{10}, 2^8, \left(\frac{7}{2}\right)^{10}, \frac{-1}{2}, 41+\frac{\sqrt{7549}}{2}, 41-\frac{\sqrt{7549}}{2}, \frac{19}{2}-\sqrt{145}, \frac{19}{2}+\sqrt{145}\Big\}$ *and* $Spec_{RD}(L^3(G')\blacktriangledown H) = \Big\{\left(\frac{-3}{2}\right)^{121}, \left(\frac{1}{2}\right)^{30}, \left(\frac{3}{2}\right)^{10}, \left(\frac{9-\sqrt{5}}{4}\right)^4, \left(\frac{13-\sqrt{5}}{4}\right)^4, \left(\frac{9+\sqrt{5}}{4}\right)^4, \left(\frac{7}{2}\right)^2, \left(\frac{13+\sqrt{5}}{4}\right)^4, \frac{-1}{2}, 41+\frac{\sqrt{7549}}{2}, 41-\frac{\sqrt{7549}}{2}, \frac{19}{2}-\sqrt{145}, \frac{19}{2}+\sqrt{145}\Big\}$*. We get* $RDE\big(L^3(G)\blacktriangledown H\big) = 263+\sqrt{7549}+2\sqrt{145} = RDE\big(L^3(G')\blacktriangledown H\big)$ *but,* $RDE\big(L^2(G)\blacktriangledown H\big) \neq RDE\big(L^2(G')\blacktriangledown H\big)$.

Theorem 4.8. *Let* G *and* G' *be* r*-regular non-cospectral graphs on* n *vertices and* H *be a* k*-regular graph of order* p*. Then* $\overline{L^m(G)}\blacktriangledown H$ *and* $\overline{L^m(G')}\blacktriangledown H$ *form RD-non-cospectral and RD-equienergetic graphs for all* $m \geq 1$.

Proof. **Case 1:** $m = 1$
By theorems 2.1, 2.2 and corollary 3.1, $Spec_{RD}(\overline{L(G)}\blacktriangledown H)$ consists of the following.

$$-\frac{1}{2}\left(\mu_j(G)+r\right), j=2,3,\ldots,n, \text{ each with multiplicity 2;}$$
$$0, \text{ with multiplicity } n(r-2);$$
$$\frac{1}{3}\left(2\mu_j(H)+\frac{1}{2}\right), \quad j=2,3,\ldots,p;$$
$$\frac{1}{3}\left(\mu_j(H)-\frac{7}{2}\right), \quad j=2,3,\ldots,p;$$

and four more numbers that are the solutions of the equation

$$\left(\left(x-\frac{2}{3}nr+r\right)\left(x-\frac{5}{6}p-\frac{2}{3}k-\frac{1}{6}\right)-\frac{9}{8}nrp\right)\cdot \left(\left(x-\frac{1}{3}nr+r\right)\left(x-\frac{1}{6}p-\frac{1}{3}k+\frac{7}{6}\right)-\frac{1}{8}nrp\right)=0, \tag{4.13}$$

obtained from equation(3.5), by replacing n and r by $\frac{nr}{2}$ and $\frac{nr}{2}-2r+1$ respectively. Proceeding in a similar manner, we obtain the RD-spectrum of $\overline{L(G')}\blacktriangledown H$. Observe that $\overline{L(G)}\blacktriangledown H$ and $\overline{L(G')}\blacktriangledown H$ are RD-non-cospectral.

Let $y_j, j=1,2,3,4$ be the solutions of equation (4.13) and $f_2(n,r,p,k)=\sum_{j=1}^{4}|y_j|$. Then using equation (1.2),

$$\begin{aligned} RDE\left(\overline{L(G)}\blacktriangledown H\right) &= 2\times\frac{1}{2}\sum_{j=2}^{n}\left|\mu_j(G)+r\right| + n(r-2)\times 0 + S_H + f_2(n,r,p,k) \\ &= \sum_{j=2}^{n}\left(\mu_j(G)+r\right) + S_H + f_2(n,r,p,k) \\ &= (n-2)r + S_H + f_2(n,r,p,k). \end{aligned}$$

By the same arguments applied above, we obtain the RD-energy of $\overline{L(G')}\blacktriangledown H$. We can notice that $RDE\left(\overline{L(G)}\blacktriangledown H\right)=RDE\left(\overline{L(G')}\blacktriangledown H\right)$.

Case 2: $m\geq 2$

From the eigenvalues of $L^m(G)$ given by (4.12) in theorem 4.7 and by theorem 2.2 and corollary 3.1, the RD-eigenvalues of $\overline{L^m(G)}\blacktriangledown H$ are as follows.

$$-\frac{1}{2}\left(\mu_j(G)+(2^m-1)r-2^{m+1}+4\right), j=2,3,\ldots,n, \text{ each with multiplicity 2;}$$
$$-\frac{1}{2}\left((2^m-2)(r-2)\right) \text{ with multiplicity } n(r-2)$$
$$-\frac{1}{2}(2^m-2^j)(r-2), \text{ with multiplicity } 2M_j, j=2,\ldots,m;$$
$$\frac{1}{3}\left(2\mu_j(H)+\frac{1}{2}\right), \quad j=2,3,\ldots,p;$$
$$\frac{1}{3}\left(\mu_j(H)-\frac{7}{2}\right), \quad j=2,3,\ldots,p;$$

and four more numbers that are the solutions of the equation obtained by replacing n and r by n_m and n_m-r_m-1 in (3.5) of corollary 3.1; and denote its solutions by $w_j, j=1,2,3,4$. Let $f_3(n,r,p,k)=\sum_{j=1}^{4}|w_j|$.

Clearly $r \geq 2$. Now, $\mu_j(G) + (2^m - 1)r - 2^{m+1} + 4 = \mu_j(G) + r + (2^m - 2)(r-2) \geq 0$, since $m \geq 2$ and $\mu_j(G) \geq -r$. Also, $(2^m - 2^j)(r-2) > 0$ for $j = 1, 2, \ldots, m$.

$$\begin{aligned} \therefore RDE(\overline{L^m(G)}\blacktriangledown H) = & \sum_{j=2}^{n} \big(\mu_j(G) + r + (2^m - 2)(r-2)\big) + \\ & n\Big(\frac{r}{2} - 1\Big)(2^m - 2)(r-2) + \\ & \sum_{j=2}^{m} M_j(2^m - 2^j)(r-2) + S_H + f_3(n, r, p, k) \\ & = (n-2)r + \frac{1}{2}\big(n(r-2) + 2\big)(r-2)(2^m - 2) + \\ & \sum_{j=2}^{m} M_j(2^m - 2^j)(r-2) + S_H + f_3(n, r, p, k). \end{aligned}$$

Similarly we obtain the RD-spectrum and its energy of $\overline{L^m(G')}\blacktriangledown H$. Clearly $RDE(\overline{L^m(G)}\blacktriangledown H) = RDE(\overline{L^m(G')}\blacktriangledown H)$. □

Example 4.5. *Let G be the Petersen graph and $G' = C_5 \square P_2$ and $H = P_2$. We have* $Spec_{RD}\big(\overline{L(G)}\blacktriangledown H\big) = \Big\{0^{10}, \big(\frac{-1}{2}\big)^9, (-2)^{10}, \frac{-3}{2}, \frac{39+\sqrt{1921}}{4}, \frac{39-\sqrt{1921}}{4}, \frac{13+\sqrt{345}}{4}, \frac{13-\sqrt{345}}{4}\Big\}$, $Spec_{RD}\big(\overline{L(G')}\blacktriangledown H\big) = \Big\{0^{10}, \Big(\frac{-3+\sqrt{5}}{4}\Big)^4, \Big(\frac{-7+\sqrt{5}}{4}\Big)^4, \Big(\frac{-3-\sqrt{5}}{4}\Big)^4, (-2)^2, \Big(\frac{-7-\sqrt{5}}{4}\Big)^4, \frac{-1}{2}, \frac{-3}{2}, \frac{39+\sqrt{1921}}{4}, \frac{39-\sqrt{1921}}{4}, \frac{13+\sqrt{345}}{4}, \frac{13-\sqrt{345}}{4}\Big\}$, $Spec_{RD}\big(\overline{L^2(G)}\blacktriangledown H\big) = \Big\{0^{30}, (-1)^{10}, \big(\frac{-3}{2}\big)^9, (-3)^{10}, \frac{-1}{2}, \frac{77+\sqrt{6649}}{4}, \frac{77-\sqrt{6649}}{4}, \frac{31+\sqrt{1329}}{4}, \frac{31-\sqrt{1329}}{4}\Big\}$ *and* $Spec_{RD}\big(\overline{L^2(G')}\blacktriangledown H\big) = \Big\{0^{30}, (-1)^{10}, \Big(\frac{-7+\sqrt{5}}{4}\Big)^4, \Big(\frac{-11+\sqrt{5}}{4}\Big)^4, \Big(\frac{-7-\sqrt{5}}{4}\Big)^4, (-3)^2, \Big(\frac{-11-\sqrt{5}}{4}\Big)^4, \frac{-1}{2}, \frac{-3}{2}, \frac{77+\sqrt{6649}}{4}, \frac{77-\sqrt{6649}}{4}, \frac{31+\sqrt{1329}}{4}, \frac{31-\sqrt{1329}}{4}\Big\}$.
Here we get $RDE\big(\overline{L(G)}\blacktriangledown H\big) = 26 + \frac{1}{2}\big(\sqrt{1921} + \sqrt{345}\big) = RDE\big(\overline{L(G')}\blacktriangledown H\big)$ *and* $RDE\big(\overline{L^2(G)}\blacktriangledown H\big) = 54 + \frac{1}{2}\big(\sqrt{6649} + \sqrt{1329}\big) = RDE\big(\overline{L^2(G')}\blacktriangledown H\big)$.

Theorem 4.9. *Let G and G′ be r-regular, non-cospectral and equienergetic graphs of order n and H be a k-regular graph of order p. Then $D_2^*(G)\blacktriangledown H$ and $D_2^*(G')\blacktriangledown H$ form RD-non-cospectral and RD-equienergetic graphs.*

Proof. By theorem 2.4 and corollary 3.1, $Spec_{RD}(D_2^*(G)\blacktriangledown H)$ consists of the following.

$$\begin{aligned} & \mu_j(G), j = 2, 3, \ldots, n, \quad \text{each with multiplicity 2;} \\ & -1 \ \text{ with multiplicity } \ 2n \\ & \frac{1}{3}\left(2\mu_j(H) + \frac{1}{2}\right), \quad j = 2, 3, \ldots, p; \\ & \frac{1}{3}\left(\mu_j(H) - \frac{7}{2}\right), \quad j = 2, 3, \ldots, p; \end{aligned}$$

and four more numbers that are the solutions of the equation obtained by replacing n and r by $2n$ and $2r + 1$ in (3.5) of corollary 3.1; and let $v_j, j = 1, 2, 3, 4$ denote its solutions. Let

$f_4(n,r,p,k) = \sum_{j=1}^{4} |v_j|$.

$$\therefore RDE(D_2^*(G)\blacktriangledown H) = 2\sum_{j=2}^{n} \left|\mu_j(G)\right| + 2n + S_H + f_4(n,r,p,k)$$
$$= 2(E(G) - r) + S_H + f_4(n,r,p,k)$$

Since $E(G) = E(G')$, we have $RDE(D_2^*(G)\blacktriangledown H) = RDE(D_2^*(G')\blacktriangledown H)$. □

Example 4.6. *Let $G = L(K_3\square P_2)$ and $G' = L(K_{3,3})$. Then G and G' are 4-regular graphs of order 9. We have $Spec(G) = \{(-2)^3, (-1)^2, 1^2, 2, 4\}$ and $Spec(G') = \{(-2)^4, 1^4, 4\}$. Clearly $E(G) = 16 = E(G')$. We get $Spec_{RD}\big(D_2^*(G)\blacktriangledown P_2\big) = \Big\{2^2, 1^4, (-1)^{22}, (-2)^6, \frac{-1}{2}, \frac{-3}{2}, \frac{43+3\sqrt{265}}{4}, \frac{43-3\sqrt{265}}{4}, \frac{13+3\sqrt{41}}{4}, \frac{13-3\sqrt{41}}{4}\Big\}$ and $Spec_{RD}\big(D_2^*(G')\blacktriangledown P_2\big) = \Big\{1^8, (-2)^8, (-1)^{18}, \frac{-1}{2}, \frac{-3}{2}, \frac{43+3\sqrt{265}}{4}, \frac{43-3\sqrt{265}}{4}, \frac{13+3\sqrt{41}}{4}, \frac{13-3\sqrt{41}}{4}\Big\}$. Here $RDE\big(D_2^*(G)\blacktriangledown P_2\big) = 44 + \frac{3}{2}\big(\sqrt{265} + \sqrt{41}\big) = RDE\big(D_2^*(G')\blacktriangledown P_2\big)$.*

5. RD-COSPECTRAL GRAPHS

In this section, we construct infinitely many non-isomorphic pairs of RD-cospectral graphs of diameter 3 as follows.

Theorem 5.10. *Let G and G' be non-isomorphic r-regular cospectral graphs on n vertices and H be any graph. Then $G\blacktriangledown H$ and $G'\blacktriangledown H$ form non-isomorphic RD-cospectral graphs.*

Proof. The result is straightforward from theorem 3.5. □

Corollary 5.2. *Let G and G' be non-isomorphic r-regular cospectral graphs on n vertices and H be any graph. Then $L^m(G)\blacktriangledown H$ and $L^m(G')\blacktriangledown H$, $m \geq 1$, form non-isomorphic RD-cospectral graphs.*

Corollary 5.3. *Let G and G' be non-isomorphic r-regular cospectral graphs on n vertices and H be any graph. Then $D_2^*(G)\blacktriangledown H$ and $D_2^*(G')\blacktriangledown H$ form non-isomorphic RD-cospectral graphs.*

Corollary 5.4. *Let G and G' be non-isomorphic r-regular cospectral graphs on n vertices and H be any graph. Then $\overline{G}\blacktriangledown H$ and $\overline{G'}\blacktriangledown H$ form non-isomorphic RD-cospectral graphs.*

Corollary 5.5. *Let G and G' be non-isomorphic r-regular cospectral graphs on n vertices and H be any graph. Then $\overline{L^m(G)}\blacktriangledown H$ and $\overline{L^m(G')}\blacktriangledown H$, $m \geq 1$, form non-isomorphic RD-cospectral graphs.*

Corollary 5.6. *Let G and G' be non-isomorphic r-regular cospectral graphs on n vertices and H be any graph. Then $\overline{D_2^*(G)}\blacktriangledown H$ and $\overline{D_2^*(G')}\blacktriangledown H$ form non-isomorphic RD-cospectral graphs.*

Theorem 5.11. *Let H and H' be non-isomorphic k-regular cospectral graphs on p vertices and G be any regular graph. Then $G\blacktriangledown H$ and $G\blacktriangledown H'$ form non-isomorphic RD-cospectral graphs.*

Proof. The result follows immediately from corollary 3.1. □

The following corollary can be directly inferred from the preceding theorem.

Corollary 5.7. *Let H and H' be non-isomorphic k-regular cospectral graphs on p vertices and G be a regular graph. Then*

(a) $G\blacktriangledown\overline{H}$ and $G\blacktriangledown\overline{H'}$ form non-isomorphic RD-cospectral graphs.
(b) $G\blacktriangledown\overline{L(H)}$ and $G\blacktriangledown\overline{L(H')}$ form non-isomorphic RD-cospectral graphs.
(c) $G\blacktriangledown L^m(H)$ and $G\blacktriangledown L^m(H')$, $m \geq 1$, form non-isomorphic RD-cospectral graphs.
(d) $G\blacktriangledown D_2^*(H)$ and $G\blacktriangledown D_2^*(H')$ form non-isomorphic RD-cospectral graphs.

6. Conclusion

Indu-Bala product, a novel graph product, has attracted many researchers in the field of domination, spectral graph theory, topological indices, metric dimension, and more. The RD-spectrum of Indu-Bala product was unknown and in our work the problem of RD-spectrum of Indu-Bala product of regular graphs is settled. In addition, using this result, we have constructed pairs of RD-equienergetic graphs as well as RD-cospectral graphs of diameter 3.

In [15] the authors constructed a pair of distance equienergetic graphs on $18 + 2k, k \geq 1$ vertices. We have generalized similar constructions to produce many pairs of RD-equienergetic graphs. In future work, we aim to extend this study to other graph classes beyond regular graphs.

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