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Some classes of distance spaces as generalized metric spaces: terminology, mappings, fixed points and applications in Theoretical Informatics

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ABSTRACT. In this paper we present a point of view on the terminology of distance spaces (names, basic notions, convergence sequence, Cauchy sequence, contraction mapping, induced order, associated metric,...) and corresponding contraction principle, and fixed point principle of increasing mappings, in such spaces. Applications to theoretical computer science are also considered.

1. INTRODUCTION AND PRELIMINARIES

1.1. **Distances and generalized metrics.** Let X be a nonempty set. For a *distance* on X we have not a precise meaning. For example, a mapping $d : X \times X \to \mathbb{R}_+$ is called a *distance function*:

- in Birkhoff [29], if $d(x,x) = 0$ for all $x \in X$ and $d(x,z) \leq d(x,y) + d(y,z)$, $\forall x,y,z \in X$;
- in Deza and Deza [50], if $d(x,x) = 0$ and $d(x,y) = d(y,x)$, for all $x,y \in X$;
- in Berinde and Choban [25], if $d(x,y) + d(y,x) = 0 \Leftrightarrow x = y$, for all $x,y \in X$.

In what follows, by a distance function on X we understand a mapping $d : X \times X \to \mathbb{R}_+$. We also consider the *axioms of a metric* d on X in the following way:

(I) For all $x \in X$, $d(x,x) = 0$.
(II) For all $x,y \in X$, if $d(x,y) = d(y,x) = 0$, then $x = y$.
(III) For all $x,y \in X$, $d(x,y) = d(y,x)$.
(IV) For all $x,y,z \in X$, $d(x,z) \leq d(x,y) + d(y,z)$.

There exist various classes of distance functions as generalized metrics on X. Some of them are listed below:

- *pseudometric* (also called *gauge*, *semimetric*; see Bourbaki [35], Dugundji [55], Engelking [57]): one drops (II) from the metric axioms;

- *quasimetric* (*écart*, *non-symmetric metric*; see Deza and Deza [50], Engelking [57], Hitzler [79], Kelley [98], Bonsangue et al. [32]): one drops (III) from the metric axioms;

- *ultrametric* (*non-archimedian metric*; see Deza and Deza [50], Hitzler [79], Ribenboim [152], Priess and Ribenboim [144], Berinde and Choban [25], Schörner [172], Arutyunov and Greshnov [13]): (IV) in the metric axioms is replaced by the more

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restrictive one

(IV') for all $x, y, z \in X$, $d(x, z) \leq \max\{d(x, y), d(y, z)\}$;

- *b-metric* (*quasimetric*, *nearmetric*, *weak metric*; see Berinde and Păcurar [26], Deza and Deza [50], Vulpe et al. [181], Bakhtin [18], Czerwik [45], Berinde [22]), Boriceanu [33], Bota et al. [34]: instead of (IV) in the metric axioms one considers

  (IV'') there exists $s \geq 1$ such that $d(x, z) \leq s\,(d(x, y) + d(y, z))$, for all $x, y, z \in X$;

- *semimetric* (Chittenden [41], Niemytzki [126], Fréchet [67], Wilson [185],...): one drops (IV) in the list of metric axioms;

- *dislocated metric* (*domain metric*, *metametric*; see Mattheus [117], Hejmej [78]): one drops (I) in the list of metric axioms;

- *partial metric* (Matthews [118, 120], Alghamdi et al. [7]):

  (1) the axiom (I) is replaced by
  $d(x, x) \leq d(x, y)$ and $d(y, y) \leq d(x, y)$, for all $x, y \in X$;
  (2) the axiom (II) is replaced by
  $d(x, x) = d(x, y) = d(y, y)$ implies $x = y$;
  (3) the axiom (IV) is replaced by
  $d(x, z) \leq d(x, y) + d(y, z) - d(y, y)$, for all $x, y, z \in X$.

- *quasi-pseudometric* (*premetric*; see Deza and Deza [50], Fréchet [67], Kasahara [96], Berinde and Choban [25]): one drops (II) and (III) in the list of metric axioms, i.e., (III) in the list of pseudometric axioms;

- *quasi-ultrametric* (Fréchet [67], Deza and Deza [50]): axiom (III) in the definition of the ultrametric is dropped;

- *dislocated quasimetric* (*dq metric*; Aage and Salunke [1], Isufati [88], Jungck and Rhoades [93], Pasicki [135]): axiom (III) in the definition of the dislocated metric is dropped.

1.2. **Generalized distances and generalized metrics: Γ-distances.** Let X be a nonempty set and Γ another nonempty set with some structure. By a Γ-*distance* we understand a mapping $d : X \times X \to \Gamma$. Here are some examples:

(1) $\Gamma = \overline{\mathbb{R}}_+ := \mathbb{R}_+ \cup \{\infty\}$. A $\overline{\mathbb{R}}_+$-distance is called *extended distance*. In case it satisfies axioms similar to the $\mathbb{R}_+$-metric, we call it an *extended metric*. Consequently we have the following corresponding notions: extended pseudometric, extended quasimetric, extended ultramentric, extended dislocated metric and so on. References: Bonsangue et al. [32], Petruşel et al. [137], Aull and Lowen [16], Rus et al. [166], Jung [92],...
(2) $\Gamma = \overline{\mathbb{R}}^m_+$. References: Ortega and Rheinboldt [128], Rus et al. [166],...
(3) $\Gamma = K$, where K is the positive cone in an ordered Banach space. References: Zabrejko [187], Proinov [145], Huang and Zhang [86], Du [53],...

(4) Γ:= an ordered set with the least element, 0. References: Hitzler [79], Kopperman [105], Kurepa [110], Seda and Hitzler [176], Turinici [183], Kabil et al. [94], Berinde [24],...
(5) For other examples, see Angelov [10], [11], Arhangel'skii [12], Asadi et al. [15], Berinde [23],

Problem 1.1. *What do we understand by a Γ-metric, Γ-pseudometric, Γ-quasimetric, Γ-ultrametric, Γ-partial metric,... ?*

References: Seda and Hitzler [176], Turinici [183], Kabil et al. [94], Hitzler and Seda [83], Iséki [87], Hitzler [79], Rutten [168], Khamsi [101], Precup [142], Marinescu [116],...

1.3. **L-spaces (M. Fréchet** [66], [67]**).** Let X be a nonempty set, $s(X) := \{(x_n) | x_n \in X, n \in \mathbb{N}\}$, $c(X) \subset s(X)$ and $Lim : c(X) \to X$ a mapping. By definition, the triple $(X, c(X), Lim)$ is an *L-space* iff:
(a) if $x_n = x$, $\forall n \in \mathbb{N}$, then $(x_n) \subset c(X)$ and $Lim(x_n) = x$;
(b) if $(x_n) \subset c(X)$ and $Lim(x_n) = x$, then all subsequences (x_{n_i}) of (x_n) are in $c(X)$ and $Lim(x_{n_i}) = x$.

We call an element of $c(X)$ a *convergent sequence* and the limit of (x_n), $Lim(x_n) = x$, is denoted by $x_n \xrightarrow{F} x$ as $n \to \infty$.

An L-space is denoted by $(X, \xrightarrow{F})$.

Problem 1.2. *Which generalized metrics are inducing an L-space structure?*

For more considerations on L-space theory we refer to Kuratowski [109], Dudley [54], Rus [164], Kasahara [97], Rus et al. [166], Filip [62],...

1.4. **The contraction principle on a complete metric space.** There exist some variants of the contraction principle, see Berinde et al. [27]. Here it is one of them (we denote, as usually, the set of fixed points of a mapping f by F_f and the n-th iterate of f by f^n).

Theorem 1.1 (Rus [162])**.** *Let (X, d) a complete metric space and let $f : X \to X$ be an l-contraction. Then we have:*
(i) There exists $x^* \in X$ such that
$$F_f = F_{f^n} = \{x^*\}, \forall n \in \mathbb{N}^*;$$
(ii) For all $x \in X$, $f^n(x) \to x^*$ as $n \to \infty$;
(iii) $d(x, x^*) \leq \Psi(d(x, f(x)))$, $\forall x \in X$, where $\Psi(t) = \dfrac{t}{1-l}$;
(iv) If (y_n) is a sequence in X such that
$$d(y_n, f(y_n)) \to 0 \text{ as } n \to \infty,$$
then $y_n \to x^*$ as $n \to \infty$;
(v) If (y_n) is a sequence in X such that
$$d(y_{n+1}, f(y_n)) \to 0 \text{ as } n \to \infty,$$
then $y_n \to x^*$ as $n \to \infty$;
(vi) If $Y \subset X$ is a closed subset such that $f(Y) \subset Y$, then $x^* \in Y$. Moreover, if in addition, Y is bounded, then
$$\bigcap_{n \in \mathbb{N}} f^n(Y) = \{x^*\}.$$

Problem 1.3. *In which generalized metric space does a similar result to Theorem 1.1 hold?*

1.5. **Fixed point results for increasing mappings.** Let X be a nonempty set. By definition, a binary relation $\leq$ on X is a *preorder* iff:

(1) $x \leq x$, for all $x \in X$;
(2) for all $x, y, z \in X$, if $x \leq y$ and $y \leq z$, then $x \leq z$.

If in addition it satisfies

(3) for all $x, y \in X$, if $x \leq y$ and $y \leq x$, then $x = y$,

then $\leq$ is called an *order relation* on X (some authors call it *partial order relation* !).

If $\leq$ is a preorder on X then we call the pair $(X, \leq)$ a *preorder set* and, if $\leq$ is an order on X, we call the pair $(X, \leq)$ an *ordered set* (also called *poset*).

In this article we follow the ordered set terminology (total ordered set, chain, minimum / maximum element, sup, infimum, inf, directed ordered set, etc.) like in Păcurar and Rus [129]. See also Davey and Priestley [47], Grätzer [71], Park [130],...

An ordered set $(X, \leq)$ is called *ω-complete* iff:

(a) $(X, \leq)$ has the minimum element 0;
(b) if $x_0 \leq x_1 \leq \dots \leq x_n \leq \dots$ is an increasing sequence in X then $\sup\{x_n | n \in \mathbb{N}\}$ exists.

A *chain* in $(X, \leq)$ is a totally ordered subset of X. An ordered set $(X, \leq)$ is called *chain complete* if every chain in X has a supremum.

A function $f : (X, \leq) \to (X, \leq)$ is increasing if for all $x_1, x_2 \in X$, $x_1 \leq x_2 \Longrightarrow f(x_1) \leq f(x_2)$.

If $(X, \leq)$ and $(Y, \leq)$ are ω-complete ordered sets, then $f : X \to Y$ is called *ω-continuous* if, for every sequence $x_0 \leq x_1 \leq \dots \leq x_n \leq \dots$ in X we have

$$f(\sup\{x_n : n \in \mathbb{N}\}) = \sup\{f(x_n) : n \in \mathbb{N}\}.$$

The following results are well known, see Fitting [63], Hitzler [79], Hitzler and Seda [83].

Theorem 1.2 (see Hitzler [79]). *Let $(X, \leq)$ be a ω-complete ordered set and $f : X \to X$ be increasing and ω-continuous. Then $x^* = \sup f^n(0)$ is the minimum (i.e., the least) fixed point of f in X.*

Theorem 1.3 (see Hitzler [79]). *Let $(X, \leq)$ be a chain-complete ordered set and $f : X \to X$ be increasing. Let $x_0 \in X$ be such that $x_0 \leq f(x_0)$.*

Then f has the minimum fixed point x^* in $\{x \in X : x_0 \leq x\}$ and, moreover, there exists the minimum ordinal α such that $x^* = f^\alpha(x_0)$.

We recall, see Hitzler [79], that the ordinal powers of a mapping $f : X \to X$ are defined as follows:

- if $\alpha + 1$ is a successor ordinal, then $f^{\alpha+1} := f(f^\alpha)$.
- if α is a limit ordinal, then $f^\alpha(x) := \sup\{f^\beta(x) : \beta < \alpha\}$.

A generalized distance d on a set X induces an order $\leq_d$ on X. For example, if d is an $\mathbb{R}_+$-quasimetric on X, one has the order $\leq_d$ defined by $x \leq_d y$ if $d(x, y) = 0$.

In our study, we shall give some other examples.

Problem 1.4. *Which conditions have to be imposed on d in order to obtain in $(X, \leq_d)$ results similar to Theorems 1.2 and 1.3?*

1.6. **Standard notions on distance spaces.** Let (X,d) be a distance space with $d : X\times X \to \mathbb{R}_+$. We consider the following concepts:

- *convergence* of a sequence $(x_n)_{n\in\mathbb{N}}$
  - *forward convergent*: there exists a unique $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$;
  - *backward convergent*: there exists a unique $x \in X$ such that $d(x, x_n) \to 0$ as $n \to \infty$;
  - *convergent*: if $(x_n)_{n\in\mathbb{N}}$ converges forward and backward.

  In the above three cases we say that x is the (forward, backward) limit of $(x_n)_{n\in\mathbb{N}}$, respectively.
- *Cauchy sequence*
  - *forward Cauchy*: if $d(x_m, x_n) \to 0$ as $n \geq m \to \infty$;
  - *backward Cauchy*: if $d(x_m, x_n) \to 0$ as $m \geq n \to \infty$;
  - *Cauchy*: if $d(x_m, x_n) \to 0$ as $n, m \to \infty$;
- *completeness* of (X,d)
  - *forward complete* if any forward Cauchy sequence is forward convergent;
  - *backward complete* if any backward Cauchy sequence is backward convergent;
  - *complete* if any Cauchy sequence is convergent;
  - *forward-backward complete* if any forward Cauchy sequence is backward convergent;
  - *backward-forward complete* if any backward Cauchy sequence is forward convergent.
- *contraction*: there exists $l \in (0,1)$ such that

$$d(f(x), f(y)) \leq ld(x,y),\ \forall x,y \in X.$$

Remark 1.1. *The above notions are defined in a similar manner in the case of* Γ*-distances, where* Γ *is an ordered L-space with the minimum element,* 0.

Remark 1.2. *Another way to induce a convergence structure on a distance space is considering the notion of* ball. *For the topologies induced by generalized metrics see Granas* [70], *Kopperman* [105], *Künzi* [107], *Agarwal et al.* [6], [5], *Aull and Lowen* [16], *Berinde and Choban* [25], *Blumenthal* [31], *Bonsangue et al.* [32], [35], *Chittenden* [41], *deGroot* [48], *Dudley* [54], *Dugundji* [55], *Engelking* [57], *Fréchet* [66], *Hausdorff* [75], *Hitzler and Seda* [82], *Hitzler and Seda* [83], *Kirk and Shahzad* [103], *Kuratowski* [109], *Reilly* [148], *Rutten* [168], *Seda and Hitzler* [175], *Skala* [179],...

2. Classes of mappings on distance spaces and fixed points

Let X be a nonempty set, $d : X \times X \to \mathbb{R}_+$ a distance and $f : X \to X$ a mapping. By definition f is:

- *nonexpansive* if $d(f(x), f(y)) \leq d(x,y)$, for any $x, y \in X$;
- *contractive* if $d(f(x), f(y)) < d(x,y)$, for any $x, y \in X$, $x \neq y$;
- *contraction* if there exists $0 < l < 1$ such that $d(f(x), f(y)) \leq ld(x,y)$, for any $x, y \in X$;
- *graphic nonexpansive* if $d(f(x), f^2(x)) \leq d(x, f(x))$, for any $x \in X$;
- *graphic contractive* if $d(f(x), f^2(x)) < d(x, f(x))$, for any $x \in X$, $x \neq f(x)$;
- *graphic contraction* if there exists $0 < l < 1$ such that $d(f(x), f^2(x)) \leq ld(x, f(x))$, for any $x \in X$;

- *continuous* (forward continuous) if the following implication holds:

$$x_n \in X, n \in \mathbb{N}, x \in X, d(x_n, x) \to 0 \text{ as } n \to \infty$$

implies

$$d(f(x_n), f(x)) \to 0 \text{ as } n \to \infty.$$

Now let (X, d) be a distance space and $f : X \to X$ an l-contraction. Regarding the fixed points of the mapping f we mention the following results.

Lemma 2.1. *If $x^*, y^* \in F_f$, then $d(x^*, y^*) = d(y^*, x^*) = 0$. If d satisfies the axiom (II), then $Card\, F_f \leq 1$.*

Lemma 2.2. *If d satisfies the axiom (IV), then for each $x \in X$ the sequence $(f^n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence. Moreover,*

$$\sum_{n=0}^{\infty} d(f^n(x), f^{n+1}(x)) \leq \frac{1}{1-l} d(x, f(x)), \text{ for any } x \in X$$

and

$$\sum_{n=0}^{\infty} d(f^{n+1}(x), f(x)) \leq \frac{1}{1-l} d(f(x), x), \text{ for any } x \in X.$$

Theorem 2.4. *Let (X, d) be a distance space and $f : X \to X$ an l-contraction. We assume that:*

(A) d satisfies axioms (II) and (IV);

(B) for any Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in (X, d) there exists a unique $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$.

Then the following hold:

(i) $F_f = F_{f^n} = \{x^*\}$, $\forall n \in \mathbb{N}$;

(ii) $d(f^n(x), x^*) \to 0$ as $n \to \infty$ and $d(x^*, f^n(x)) \to 0$ as $n \to \infty$, for all $x \in X$;

(iii) $d(x, x^*) \leq \frac{1}{1-l} d(x, f(x))$ and $d(x^*, x) \leq \frac{1}{1-l} d(f(x), x)$, for all $x \in X$;

(iv) $x_n \in X, n \in \mathbb{N}, d(x_n, f(x_n)) \to 0$ as $n \to \infty$ implies $d(x_n, x^*) \to 0$ as $n \to \infty$ and $x_n \in X, n \in \mathbb{N}, d(f(x_n), x_n) \to 0$ as $n \to \infty$ implies $d(x^*, x_n) \to 0$ as $n \to \infty$;

(v) $x_n \in X, n \in \mathbb{N}, d(x_{n+1}, f(x_n)) \to 0$ as $n \to \infty$ implies $d(x_n, x^*) \to 0$ as $n \to \infty$ and $x_n \in X, n \in \mathbb{N}, d(f(x_n), x_{n+1}) \to 0$ as $n \to \infty$ implies $d(x^*, x_n) \to 0$ as $n \to \infty$.

Proof. $(i), (ii)$ From assumption (A), by Lemma 2.1, it follows that $CardF_f \leq 1$. From assumption (B), by Lemma 2.2, it follows that for any $x \in X$ there exists $x^*(x)$ such that

$$d(f^n(x), x^*(x)) \to 0, \text{ as } n \to \infty.$$

On the other hand

$$d(f^{n+1}(x), f(x^*(x))) \leq l d(f^n(x), x^*(x)) \to 0 \text{ as } n \to \infty.$$

Since the limit is unique, it follows that $x^*(x) = f(x^*(x))$, i.e., $x^*(x) \in F_f$. But $CardF_f \leq 1$, so $F_f = \{x^*\}$ and $d(f^n(x), x^*) \to 0$ as $n \to \infty$, for all $x \in X$. Since x^* is a fixed point of f and f is a contraction,

$$d(x^*, f^n(x)) = d(f^n(x^*), f^n(x)) \leq l^n d(x^*, x) \to 0 \text{ as } n \to \infty.$$

But $F_f \subset F_{f^n}$, for all $n \in \mathbb{N}^*$, and f^n is a contraction, so conclusion (ii) follows.

(iii) We have that

$$d(x, x^*) \leq d(x, f(x)) + d(f(x), x^*) \leq d(x, f(x)) + l d(x, x^*)$$

and similarly the other relation, so the conclusion follows.

An alternative proof of (iii) is the following:

$$\begin{aligned} d(x,x^*) &\le d(x,f(x)) + d(f(x),f^2(x)) + \cdots + d(f^n(x),x^*) \\ &\le \sum_{k=0}^{\infty} d(f^k(x),f^{k+1}(x)) + d(f^n(x),x^*). \end{aligned}$$

Letting $n \to \infty$ it follows that

$$d(x,x^*) \le \sum_{k=0}^{\infty} d(f^k(x),f^{k+1}(x)) \le \frac{1}{1-l} d(x,f(x)).$$

This proof for (iii) is working in the case of generalized contractions.

Conclusion (iv) follows from (iii).

(v) Since f is a quasicontraction, we have that:

$$\begin{aligned} d(x_{n+1},x^*) &\le d(x_{n+1},f(x_n)) + d(f(x_n),x^*) \\ &\le d(x_{n+1},f(x_n)) + l d(x_n,x^*) \\ &\le \cdots \\ &\le d(x_{n+1},f(x_n)) + l d(x_n,f(x_{n-1})) + l^2 d(x_{n-1},f(x_{n-2})) + \\ &\quad + \cdots + l^n d(x_1,f(x_0)) + l^{n+1} d(x_0,x^*), \end{aligned}$$

which converges to 0 by the Cauchy-Toeplitz Lemma (see Rus [162, 167]). □

In order to present the next theorem we need the following notion (see for example Rus [164, 158, 163]).

Let $(X, \overset{F}{\to})$ be an L-space and $f : X \to X$ a weakly Picard mapping (WPM). For each $x^* \in F_f$ we consider the subset $X_{x^*} := \{x \in X | f^n(x) \to x^* \text{ as } n \to \infty\}$. Then $f(X_{x^*}) \subset X_{x^*}$ and $X = \bigcup_{x^* \in F_f} X_{x^*}$ is a partition of X. We call this partition the *fixed point partition* of X corresponding to f.

In the case of graphic contractions we have the following fixed point result:

Theorem 2.5. *Let (X,d) be a distance space and $f : X \to X$ a graphic l-contraction. We assume that:*

(A) d satisfies the metric axioms (II) and (IV);
(B) for each Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in X, there exists a unique x such that $d(x_n,x) \to 0$ as $n \to \infty$;
(C) f is continuous.

Then the following hold:

(i) $F_f = F_{f^n} \neq \emptyset$;
(ii) for each $x \in X$, there exists $f^\infty(x) \in F_f$ such that

$$d(f^n(x), f^\infty(x)) \to 0 \text{ as } n \to \infty;$$

(iii) $d(x,f^\infty(x)) \le \frac{1}{1-l} d(x,f(x))$, for all $x \in X$;
(iv) for each $x^* \in F_f$, the following implication holds:

$$x_n \in X_{x^*}, n \in \mathbb{N}, d(x_n,f(x_n)) \to 0, n \to \infty \Rightarrow x_n \to x^*, n \to \infty.$$

If, in addition, d satisfies the metric axiom (III) and $l < \frac{1}{3}$, then:

(v) for each $x^* \in F_f$, $d(f(x),x^*) \le \frac{l}{1-2l} d(x,x^*), \forall x \in X_{x^*}$,

i.e., $d(f(x),f^\infty(x)) \le \frac{l}{1-2l} d(x,f^\infty(x)), \forall x \in X$;

(vi) for each $x^* \in F_f$, the following implication holds:

$$x_n \in X_{x^*}, n \in \mathbb{N}, d(x_{n+1}, f(x_n)) \to 0 \text{ as } n \to \infty$$

implies

$$x_n \to x^* \text{ as } n \to \infty.$$

Proof. Since f is a graphic l-contraction, there exists $l \in (0,1)$ such that

$$d(f(x), f^2(x)) \leq l d(x, f(x)), \forall x \in X. \tag{2.1}$$

Hence, by (2.1), we obtain that

$$d(f^n(x), f^{n+1}(x)) \leq l^n d(x, f(x)), \forall x \in X, n \geq 1. \tag{2.2}$$

So, by (IV) and (2.2) we get

$$d(f^n(x), f^{n+p}(x)) \leq l^n(1 + l + \cdots + l^{p-1}) d(x, f(x)) < \frac{l^n}{1-l} d(x, f(x)),$$

which implies that $(f^n(x))$ is a Cauchy sequence.

Therefore, by (B), it follows that, for any $x \in X$ there exists a unique $\overline{x} = \overline{x}(x) \in X$ such that $d(f^n(x), \overline{x}) \to 0$ as $n \to \infty$.

By (C), $\left(f^{n+1}(x)\right)$ converges to $f(\overline{x})$, i.e., $d(f^{n+1}(x), f(\overline{x})) \to 0$ as $n \to \infty$, which, in view of (B), shows that $f(\overline{x}) = \overline{x}$, that is, $\overline{x} \in F_f$.

By denoting $f^\infty(x) := \overline{x}(x)$, *(i)* and *(ii)* follow.

To prove *(iii)*, we first note that by (2.2) one has

$$\sum_{n=0}^{\infty} d(f^n(x), f^{n+1}(x)) \leq d(x, f(x)) \cdot \sum_{n=0}^{\infty} l^n = \frac{1}{1-l} d(x, f(x)). \tag{2.3}$$

On the other hand

$$d(x, f^\infty(x)) \leq d(x, f(x) + d(f(x), f^2(x)) + \cdots + d(f^n(x), f^{n+1}(x))$$
$$+ d(f^{n+1}(x), f^\infty(x)) \leq \sum_{n=0}^{\infty} d(f^n(x), f^{n+1}(x)) + d(f^{n+1}(x), f^\infty(x))$$

and hence, by letting $n \to \infty$ in the previous inequality and using (2.3), we get exactly *(iii)*.

(iv) follows by taking $x := x_n$ in *(iii)* and using the fact that $x^* = f^\infty(x_n)$.

(v) Since $x \in X_{x^*}$, we have $f^n x \to x^*$ as $n \to \infty$, that is, $d(f^n x, x^*) \to 0$ as $n \to \infty$. Next,

$$\begin{aligned} d(f(x), x^*) &\leq d(f(x), f^2(x)) + d(f^2(x), x^*) \\ &\leq l \cdot d(x, f(x)) + d(f^2(x), x^*) \ldots \\ &\leq (l + l^2 + \cdots + l^n) \cdot d(x, f(x)) + d(f^{n+1}(x), x^*) \end{aligned}$$
$$< \frac{l}{1-l} \cdot d(x, f(x)) + d(f^{n+1}(x), x^*) \leq \frac{l}{1-l} \cdot [d(x, x^*) + d(x^*, f(x)]$$
$$+ d(f^{n+1}(x), x^*).$$

Since d is symmetric, we get

$$d(f(x), x^*) \leq \frac{l}{1-2l} \cdot d(x, x^*) + \frac{1-l}{1-2l} \cdot d(f^{n+1}(x), x^*),$$

which, by letting $n \to \infty$, yields exactly the estimate in *(v)*.

(vi) As $x_1 \in X_{x^*}$, we have $f^n(x_1) \to x^*$ as $n \to \infty$, that is, $d(f^n(x_1), x^*) \to 0$ as $n \to \infty$. Next,

$$d(x_{n+1}, x^*) \leq d(x_{n+1}, f(x_n)) + d(f(x_n), x^*) \leq d(x_{n+1}, f(x_n))$$
$$+ d(f(x_n), f^2(x_{n-1})) + d(f^2(x_{n-1}, x^*) \leq d(x_{n+1}, f(x_n))$$

$$+ld(x_n, f(x_{n-1})) + d(f^2(x_{n-1}), f^3(x_{n-2})) + d(f^3(x_{n-2}, x^*)$$
$$\leq \cdots \leq d(x_{n+1}, f(x_n)) + ld(x_n, f(x_{n-1})) + l^2 d(x_{n-1}), f(x_{n-2}))$$
$$+ \cdots + l^{n-1} d(x_2, f(x_1)) + d(f^n(x_1), x^*).$$

By using the Cauchy-Toeplitz Lemma (see Rus [162, 167]), the right hand side of the above inequality converges to 0, which ends the proof. □

Remark 2.3. *We remark that the axiomatic system of the dislocated metric spaces, partial metric spaces, ultrametric spaces contains the axioms (II) and (IV). If we consider the* standard notions *in each space, then the completeness implies the condition (B) from Theorems 2.4 and 2.5. So, we have the* Saturated contraction principle *and* Saturated contraction graphic principle *in each of these spaces (see Mattheus* [117], *Rus* [163], *Filip* [62], *Park* [131], *Rus* [162], *Petruşel and Rus* [140]*).*

In order to obtain similar fixed point results in a quasimetric space, we need the following additional condition:

(Q) *If $d(x_n, x) \to 0$ as $n \to \infty$, then $d(x_n, y) \to d(x, y), \forall y \in X$, i.e., $d(\cdot, x) : X \to X$ is continuous.*

Remark 2.4. *In the case of a partial metric space (X, p) one should consider different notions of convergence and completeness, as follows.*

The partial metric p induces on X the following metric:

$$d_p^s(x, y) := 2p(x, y) - p(x, x) - p(y, y), \forall x, y \in X.$$

By definition,

$$x_n \xrightarrow{p} x \text{ as } n \to \infty \Leftrightarrow x_n \xrightarrow{d_p^s} x \text{ as } n \to \infty$$

and

$$(X, p) \text{ is complete } \Leftrightarrow (X, d_p^s) \text{ is complete.}$$

In this case, the standard notion of convergence on (X, p) may be denoted by $\xrightarrow{(0)}$.

For more considerations see Matthews [118, 119], *Rus* [159], *Agarwal et al.* [5], *Filip* [62], *Hitzler* [79], *Pasicki* [133],*...*

Remark 2.5. *Some classes of distances on a set X induce an order relation on X. For example, if (X, q) is a quasimetric space, then q induces on X the following order relation:*

$$x \leq_q y \Leftrightarrow q(x, y) = 0.$$

With respect to this order relation, we have the following variant of a fixed point theorem of Rutten [168].

Theorem 2.6. *Let (X, q) be a forward complete quasimetric with the (Q) property. Let $f : X \to X$ be a mapping such that:*

(a) f is continuous w.r.t. $\xrightarrow{q}$;
(b) f is nonexpansive;
(c) there exists $x_0 \in X$ such that $x_0 \leq_q f(x_0)$.

Then f has a minimum fixed point x^* above x_0, i.e., x^* is the minimum element in $(F_f \cap \{x \in X | x_0 \leq x\}, \leq_q)$.

The proof of this variant is similar to the proof of Rutten's theorem.

Remark 2.6. *Some authors (Doitchinov* [51], *Reilly* [150], *Romaguera et al.* [154],*...) are working with a strong notion of quasimetric. Instead of axiom (II), the following axiom is considered:*

$$(II')\ q(x, y) = 0 \Leftrightarrow x = y.$$

In this case, $x \leq_q y \Leftrightarrow x = y$.

For other notions of quasimetric see Dudley [54].

3. Extending distance spaces

Let X be a nonempty set. A mapping $d : X \times X \to [0, +\infty]$ is by definition an *extending distance*. In this case, (X, d) is called an *extending distance space*. An extending distance is an *extending metric* if d satisfies the following axioms:

(eI) $d(x, x) = 0, \forall x \in X$;
(eII) $x, y \in X, d(x, y) = d(y, x) = 0 \Rightarrow x = y$;
(eIII) $d(x, y) = d(y, x), \forall x, y \in X$;
(eIV) $x, y, z \in X, d(x, z) < +\infty, d(z, y) < \infty$
$\Rightarrow d(x, y) \leq d(x, z) + d(z, y)$.

As in the case of distance spaces, we define some classes of extending distance spaces such as: extending pseudometric, extending quasimetric, extending ultrametric, extending b-metric, extending semimetric, extending dislocated metric, extending partial metric spaces, and so on.

We have a similar situation in the case of standard notions. For example if (X, d) is an extending distance space, then a mapping $f : X \to X$ is an l-contraction if $0 < l < 1$ and the following implication holds:

$$x, y \in X, d(x, y) < +\infty \;\Rightarrow\; d(f(x), f(y)) \leq l d(x, y).$$

For the saturated contraction principle and the saturated graphic contraction principle in an extending metric space see Petruşel et al. [137]. For the contraction principle in an extending quasimetric space see Seda [174], Romaguera et al. [154], Romaguera and Valero [156],...

For the case of the extending b-metric spaces see Cobzaş and Czerwik [43].

Next, we present a variant of each of the fixed point results from Section 2. In this respect we need the following notion.

By definition an extending distance is a *strong extending distance* if it satisfies the conditions:

(i) $d(x, x) < +\infty, \forall x \in X$;
(ii) if $d(x, y) < +\infty$ for $x, y \in X$, then $d(y, x) < +\infty$.

Example 3.1. *Let $(X, d_\lambda), \lambda \in \Lambda$ be a family of distance spaces such that $X_\lambda \cap X_\mu = \emptyset$ for $\lambda \neq \mu$, $\lambda, \mu \in \Lambda$. Denote $X := \bigcup_{\lambda \in \Lambda} X_\lambda$ and let $d : X \times X \to [0, +\infty]$ be defined by*

$$d(x, y) := \begin{cases} d_\lambda(x, y), \text{ if } x, y \in X_\lambda, \lambda \in \Lambda \\ +\infty, \text{ if } x \in X_\lambda, y \in X_\mu, \lambda \neq \mu, \lambda, \mu \in \Lambda. \end{cases}$$

Then (X, d) is an extending distance space.

Further on we will use the following lemmas.

Lemma 3.3. *Let (X, d) be a strong extending distance space which satisfies the axiom (eIV) (or the corresponding axiom in the case of ultrametric and of b-metric).*

Then there exists a partition of X, $X = \bigcup_{\lambda \in \Lambda} X_\lambda$, such that $d_\lambda := d|_{X_\lambda \times X_\lambda}$ is a distance, for each $\lambda \in \Lambda$. Moreover:

(1) (X, d) is forward complete if and only if (X_λ, d_λ) is forward complete for each $\lambda \in \Lambda$;
(2) (X, d) is backward complete if and only if (X_λ, d_λ) is backward complete for each $\lambda \in \Lambda$;
(3) (X, d) satisfies condition (A) if and only if (X_λ, d_λ) satisfies condition (A) for each $\lambda \in \Lambda$;
(4) (X, d) satisfies condition (B) if and only if (X_λ, d_λ) satisfies condition (B) for each $\lambda \in \Lambda$.

Lemma 3.4. *Let (X, d) be a strong extending distance space, which has the canonical partition $X = \underset{\lambda\in\Lambda}{\cup} X_\lambda$, and $f : X \to X$ an l-contraction. If there exists $x \in X_\lambda$ such that $d(x, f(x)) < +\infty$, then $f(X_\lambda) \subset X_\lambda$.*

Let $\tilde{\Lambda} := \{\lambda \in \Lambda | f(X_\lambda) \subset X_\lambda\}$. Based on the above lemmas and the fixed point results in Section 2, we obtain the following theorems.

Theorem 3.7. *Let (X, d) be a strong extending distance space which satisfies conditions (A) and (B) and $f : X \to X$ is an l-contraction.*

Then the conclusions of Theorem 2.4 hold in each $X_{\tilde{\lambda}}, \tilde{\lambda} \in \tilde{\Lambda}$.

Theorem 3.8. *Let (X, d) be a strong extending distance space which satisfies conditions (A) and (B) and $f : X \to X$ a continuous graphic l-contraction. Let $\tilde{X} := \{x \in X | d(x, f(x)) < +\infty\}$.*

Then $f(\tilde{X}) \subset \tilde{X}$ and the conclusions of Theorem 2.5 hold in $\tilde{X}$.

4. Cone distance spaces

Let $(\mathbb{B}, +, \mathbb{R}, \|\cdot\|, \leq)$ be an ordered Banach space and $\mathbb{K}$, the cone of positive elements of $\mathbb{B}$, i.e., $\mathbb{K} := \{x \in \mathbb{B} | x \geq 0\}$. In this section we assume that $\mathbb{K}$ is a normal cone (see Eisenfeld and Lakshmikantham [56], Krasnoselskii [106], Zabrejko [187], Petruşel et al. [136]).

Let X be a nonempty set and d be a Γ-distance on X, where $\Gamma := \mathbb{K} = (\mathbb{K}, +, \mathbb{R}, \xrightarrow{\|\cdot\|}, \leq)$. We adopt the same notatios, *(I)-(IV)*, for the corresponding axioms of a cone metric, like in the case of the metric in Section 1.

We call d a cone distance and (X, d) a cone distance space. Similarly to Section 2, we define the standard notions of convergence, completeness, some classes of cone distances (such as cone pseudometric, cone quasimetric, cone ultrametric, cone semimetric, cone dislocated metric, cone partial metric) and some classes of mappings (nonexpansive, contractive, graphic nonexpansive, graphic contractive).

The notion of *contraction* is more problematic in this case.

Let (X, d) be a cone distance space and $f : X \to X$ a mapping. Some authors (see Huang and Zhang [86], Proinov [145], Abdeljawad and Rezapour [2], Du [53],...) define the contraction as follows: f is a contraction if there exists $0 < l < 1$ such that

$$d(f(x), f(y)) \leq l d(x, y), \forall x, y \in X.$$

A proper definition is the following one (see Zabrejko [187], Petruşel et al. [136], Ortega and Rheinboldt [128], Rus and Şerban [167], Eisenfeld and Lakshmikantham [56],...):

Definition 4.1. *Let (X, d) a cone distance space ($\mathbb{K} \subset \mathbb{B}$). A mapping $f : X \to X$ is a contraction if there exists a bounded linear increasing mapping $Q : \mathbb{B} \to \mathbb{B}$ such that:*

(a) $\rho(Q) < 1$, *where $\rho(Q)$ is the spectral radius of Q;*

(b) $d(f(x), f(y)) \leq Q(d(x, y)), \forall x, y \in X.$

In a similar way we define the graphic Q-contraction. For the properties of bounded linear mappings with spectral radius less than 1 see Gohberg et al. [69], Rus et al. [166], Zabrejko [187],...

We have the following fixed point results.

Lemma 4.5. *If f is a Q-contraction and $x^*, y^* \in F_f$, then $d(x^*, y^*) = d(y^*, x^*) = 0$. If in addition d satisfies axiom (II), then $card F_f \leq 1$.*

Proof. From (a) in Definition 4.1, it follows that $Q^n(y) \to 0$ as $n \to \infty$, $\forall y \in K$. Let $x^*, y^* \in F_f$. Then

$$d(x^*, y^*) = d(f^n(x^*), f^n(y^*)) \leq Q^n(d(x^*, y^*)) \to 0$$

and similarly for $d(y^*, x^*)$. □

Lemma 4.6. *If f is a Q-contraction and d satisfies the axiom (IV), then for each $x \in X$ the sequence $(f^n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence.*

Proof. Denote by $\mathcal{B}(X)$ the set of all linear bounded operators on X. $\mathcal{B}(X)$ is a Banach algebra and, for $Q \in \mathcal{B}(X)$,

$$\rho(Q) := \lim_{n\to\infty} \|Q^n\|^{1/n}$$

is the *spectral radius* of Q. It is well known that, if $\rho(Q) < 1$, then the series $\sum\limits_{n=0}^{\infty} Q^n$ is absolutely convergent, $I - Q$ is invertible in $\mathcal{B}(X)$ and

$$\sum_{n=0}^{\infty} Q^n = (I - Q)^{-1}.$$

Now, to prove our Lemma, let us denote $x_{n+1} = f^n(x)$, $x \in X$. Then, by the contraction condition *(b)*, we have

$$d(x_n, x_{n+1}) = d(f(x_{n-1}, f(x_n)) \leq Q(d(x_{n-1}, x_n)).$$

Since Q is increasing, we obtain that

$$d(x_n, x_{n+1}) \leq Q^n(d(x_0, x_1)),\ \forall n \geq 1. \tag{4.4}$$

Using (4.4), we obtain

$$d(x_n, x_{n+p}) \leq \sum_{k=n}^{n+p-1} Q^k(d(x_0, x_1)) = S_{n+p} - S_{n-1},\ n, p \in \mathbb{N}, \tag{4.5}$$

where we denoted by S_n the partial sum of the series $\sum\limits_{k=0}^{\infty} Q^k(d(x_0, x_1))$.

By letting $n \to \infty$ in (4.5), we obtain that (x_n) is a Cauchy sequence. □

Theorem 4.9. *Let (X, d) be a cone distance space and $f : X \to X$ a Q-contraction. We assume that:*

(A) *d satisfies axioms (II) and (IV);*
(B) *for any Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in (X, d) there exists a unique $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$.*

Then, the analogue conclusions to those in Theorem 2.4 also hold.

Proof. The proof is similar to the proof of Theorem 2.4 and we omit it. □

Theorem 4.10. *Let (X, d) be a cone distance space and $f : X \to X$ a graphic Q-contraction. We assume that:*

(A) *d satisfies axioms (II) and (IV);*
(B) *for any Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in (X, d) there exists a unique $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$;*
(C) *f is continuous.*

Then the analogue conclusions to those in Theorem 2.5 hold.

Proof. The proof is similar to the proof of Theorem 2.5. □

For other related results in the particular case of cone metric spaces we refer to Cvetković and Rakočević [44] and references cited there.

5. Cone b-metric spaces

Let (X, d) be a cone b-metric space and $f : X \to X$ a mapping. We first consider, like in Huang and Xu [84, 85], Huang and Zhang [86], Proinov [145], Abdeljawad and Rezapour [2], Du [53], the simplest definition of a contraction in this setting: f is a contraction if there exists a numerical constant $0 < l < 1$ such that

$$d(f(x), f(y)) \leq l d(x, y), \forall x, y \in X.$$

The following theorem is a sample result of this kind, adapted after Huang and Xu [84, 85]

Theorem 5.11. *Let (X, d) be a complete cone b-metric space with $s \geq 1$ and $f : X \to X$ an l-contraction.*

Then the following hold:

(i) $F_f = F_{f^n} = \{x^*\}, \forall n \in \mathbb{N}$;
(ii) $f^n(x) \to 0$ *as* $n \to \infty$, *for all* $x \in X$;
(iii) $d(x, x^*) \leq \dfrac{1}{1-l} d(x, f(x))$, *for all* $x \in X$;
(iv) $x_n \in X, n \in \mathbb{N}, d(x_n, f(x_n)) \to 0$ *as* $n \to \infty$ *implies* $d(x_n, x^*) \to 0$ *as* $n \to \infty$;
(v) $x_n \in X, n \in \mathbb{N}, d(x_{n+1}, f(x_n)) \to 0$ *as* $n \to \infty$ *implies* $d(x_n, x^*) \to 0$ *as* $n \to \infty$.

Proof. We proceed in a similar manner to the proof of Theorem 2.4. □

More general results, similar to that in Theorem 4.9 and Theorem 4.10, could be established by considering the concept of contraction introduced in Definition 4.1.

6. Applications to Theoretical Informatics: fixed point as a definition tool

The following facts are well known.

Let (X, τ) be a topological space. Then there exists a generalized metric d on X such that the topology induced by d on X, $\tau_d = \tau$ (Kopperman [105], Bonsangue et al. [32], Flagg and Kopperman [64], Hitzler [79], Seda and Hitzler [175], Künzi [107],...).

Each ordered set and each graph can be interpreted as a generalized metric space, see Jawhari et al. [91], Quilliot [146], Abu-Sbeih and Khamsi [3], Grätzer [71], Khamsi [101], Păcurar and Rus [129], Park [130], Rus [164], Rutten and Turi [169], Baranga [20],...

Many categories can be interpreted as categories of generalized metric spaces, see Heitzig [77], Barr and Wells [21], Mac Lane [114], Rus [165], America and Rutten [9], Baranga [19], Blass [30], Ésik [59], Jawhari et al. [91], Lawvere [111], Rutten and Turi [169],...)

And now some words on the applications of generalized metric spaces to theoretical informatics.

Fixed Point Theory provides some important tools in various areas of theoretical computer science. One such area is the theory of logic programming languages, which comprises the study of their syntax and of their semantics.

Logic programs have a natural semantics, called their *declarative* semantics, which is usually represented by means of models, in the sense of mathematical logic. One is interested in selecting the right model for a program, i.e., a model which reflects the intended meaning of the programmer and relates this meaning to what the program can compute.

A common way to select the appropriate models for a logic program is to consider various associated operators, called *semantic operators*, defined on spaces of interpretations (or valuations) determined by the program. One then studies the fixed points of these operators, leading to the fixed-point semantics of the program in question.

In fact, fixed points of certain operators (and of functors) play an important role in theoretical computer science, wherever recursion and self-reference are encountered, as it is the case of conventional programming language semantics, such as the denotational semantics of functional and imperative programs. In such contexts, the involved operators are usually acting on a complete lattice and are monotonic, hence the existence of fixed points is obtained routinely via Knaster-Tarski fixed theorem.

In contrast to the above situation, when we are in the case of semantics of logic programs, by introducing negation, which is desirable from the point of view of expressiveness and enhanced syntax, then the corresponding operators associated with logic programs are no more monotonic and there is a need for metric or order-metric fixed point theorems to get the desired conclusions.

For more considerations, see Hitzler [79], Rutten and Turi [169], Frigon [68], Hitzler and Seda [80]-[83], Khamsi et al. [99], Mattheus and Bukatin [120], O'Neill [127], Priess-Crampe and Ribenboim [143], Alghamdi et al. [8], Romaguera et al. [154], Aceto et al. [4]; see also Bugajewski et al. [39], Bukatin et al. [40], Matthews [121], Plotkin [141], Romaguera and Schellekens [155], Schellekens [171], Smale [180],...

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