CREAT. MATH. INFORM.
Volume **34** (2025), No. 1,
Pages 99-111

Online version at https://creative-mathematics.cunbm.utcluj.ro/
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2025.01.10

Leverage Centrality in Corona Product of Cycle with some Graphs

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ABSTRACT. Centrality measures are used to identify highly central nodes in many diverse kinds of networks. The different perspectives of a particular node are studied under different indices, in which the leverage centrality is unique among existing measures because it measures the extent of the connectivity of a node relative to the connectivity of its nearest neighbors. In this paper, we integrate the concept of leverage centrality with the operation of corona product of two graphs and especially the leverage analysis of corona product of cycle with null, complete and path graphs are investigated. Even though the leverage centrality of all the vertices of a cycle are zero, we get surprising results when we combine cycle with null, complete and path graphs through this operation. Understanding leverage centrality in this specific graph product allows for the optimization of network design by identifying key nodes that influence network robustness and efficiency, particularly in systems modeled by cyclic, null, complete, and path structures.

1. INTRODUCTION

Leverage centrality is a novel centrality measure proposed by Joyce et al. in 2010 as a means to analyze connections within the brain. A centrality measure assigns a centrality (importance) value to each node in the network [4]. The leverage centrality of a node in a network is determined by the extent to which its direct neighbors rely on that node for information [1]. By considering neighbors degrees, leverage centrality gives different information about the connectivity of a vertex in contrast with simple degree centrality [5]. Even though leverage is derived from degree centrality, there is an essential difference between the two. In fact, a high degree node with high degree nearest neighbors will probably have a low leverage [8]. In the mathematical development of this centrality, the leverage centralities of complete multipartite graphs and the cartesian product of paths were investigated by Sharma, Vargas, Waldron, Flórez, and Narayan [12]. Leverage centrality of knight's graphs and cartesian products of regular graphs and path powers were investigated by Roger Vargas, Jr. et al.[13]. Leverage centrality analysis of some infrastructure networks were determined by Murat Erşen Berberler [1]. Hence we are motivated to analyze the leverage centrality in the corona product of two graphs for the further development of the theory. The corona operation duplicates the second graph by the cardinality of the first graph and connects each copy to each corresponding point in the first graph [6]. For the comparative study, here we selected the null graph which is disconnected and the complete graph which is connected, but both are of leverage zero for all the vertices. Now the path is connected with three distinct leverage centralities. We found that there are two distinct leverage centralities in the corona product of cycle with the null and the complete graph. As the operation is not commutative in general, the reverse product is also considered and got distinct centralities. The case with path is different, as the path itself has three distinct leverage centralities.

Received: 14.03.2024. In revised form: 24.07.2024. Accepted: 31.07.2024
2020 *Mathematics Subject Classification.* 05C90, 94C15.
Key words and phrases. *Leverage center, null leverage graph, regular graph, cyclic caterpillar.*

The findings can be directly applied to practical scenarios in telecommunications, transportation, and social networks, where optimized node placement and enhanced network reliability are critical, demonstrating the real-world applicability of leverage centrality analysis in diverse fields.

The structure of the paper is as follows. The paper has four major sections in which the first describes the motivation for the present study, basic definitions and some propositions on leverage centrality. Second section has three subsections which are dedicated to the leverage centrality analysis of corona product of cycle with null, complete and path graphs respectively. In the third section, we illustrate our results with examples and practical applications. The last section includes conclusion with the scope of future research.

We begin with some basic definitions that are essential for our study.

Definition 1.1. [3] *The number of edges incident on a vertex v is called the degree of the vertex v. We denote it by deg(v).*

Definition 1.2. [3] *An open walk in which no vertex appears more than once is called a path. A path on n vertices is denoted by P_n.*

Definition 1.3. [3] *A closed walk in which no vertex (except the initial and the final vertex) appears more than once is called a cycle. A cycle on n vertices is denoted by C_n.*

Definition 1.4. [3] *A graph without any edges is called a null graph. A null graph on m vertices is denoted by N_m.*

Definition 1.5. [3] *A simple graph in which there exists an edge between every pair of vertices is called a complete graph. A complete graph on n vertices is denoted by K_n.*

Now, we define the leverage centrality of a node v and the leverage center of a graph G are as follows:

Definition 1.6. [12] *Leverage centrality is a measure of the relationship between the degree of a given node v and the degree of each of its neighbors v_i averaged over all neighbors N_v and is defined as:*

$$l(v) = \frac{1}{deg(v)} \sum_{v_i \in N_v} \frac{deg(v) - deg(v_i)}{deg(v) + deg(v_i)}$$

Leverage is defined on the interval $(-1, 1)$, making inter- and intra-network comparisons straightforward [8].

In one of our earlier works, we defined the following concepts:

Definition 1.7. [11] *The leverage center of a graph G is defined as the set of nodes having the highest leverage centrality in the graph.*

Definition 1.8. [11] *Unicentric leverage graphs are graphs with unique leverage centers.*

Definition 1.9. [11] *Bicentric leverage graphs are graphs with exactly two leverage centers.*

Definition 1.10. [10] *A null leverage graph is a graph with all the vertices are of leverage zero.*

Definition 1.11. [14] *The corona product $G \odot H$ of two graphs G and H is defined as the graph obtained by taking one copy of G and $|V(G)|$ copies of H and joining the i^{th} vertex of G to every vertex in the i^{th} copy of H.*

Remark 1.1. [14] *If $|V(G)| = n$ and $|E(G)| = q$, we say that G is an (n, q) graph. If G is an (n, q) graph and H is an $(m, q\prime)$ graph, then $|V(G \odot H)| = n + nm$ and $|E(G \odot H)| = q + nq\prime + nm$. $G \odot H$ is connected if and only if G is connected.*

Remark 1.2. *In this paper, we use the following notations: The i^{th} copy of H is denoted by H_i, $1 \leq i \leq n$. The vertices of a graph G of order n be labeled as $\{u_1, u_2, ..., u_n\}$ and that of a graph H of order m be labeled as $\{v_1, v_2, ..., v_m\}$. Then in the graph $G \odot H$, the vertex v_k in H_i is denoted by $v_{i,k}$, $1 \leq i \leq n$ and $1 \leq k \leq m$. Similarly in $H \odot G$, the vertex u_k in G_i is denoted by $u_{i,k}$, $1 \leq i \leq m$ and $1 \leq k \leq n$. Also, $N(v)$ represents the neighborhood set of v.*

1.1. **Some Basic Propositions on Leverage Centrality.**

Proposition 1.1. [12] *Let G be a graph with n vertices. For any vertex v, $|l(v)| \leq 1 - \frac{2}{n}$. Furthermore, these bounds are tight in the cases of stars and complete graphs.*

Proposition 1.2. [12] *For any graph G, $\sum_{v \in G} l(v) \leq 0$.*

Proposition 1.3. [12] *$l(v) = 0$ for every vertex $v \in G$ if and only if G is a regular graph.*

Theorem 1.1. [12] *In a graph G of order n, the maximum number of vertices with positive leverage centrality is $n - 1$.*

2. Main result

2.1. **Leverage Centrality in Corona Product of Cycle with Null graph.**

2.1.1. *Leverage Centrality in C_n and N_m.* Firstly we analyze the leverage centrality of nodes in the component graphs C_n and N_m. Since C_n is 2-regular, the leverage centrality of all the vertices is zero by proposition 1.3. In a null graph, since there are no connections we have that the leverage of all the vertices is zero. Now we narrate the detailed analysis of leverage centrality of vertices in the corona product of C_n with N_m.

2.1.2. *Corona product $C_n \odot N_m$ (Thorn-regular cyclic caterpillars).*

Definition 2.12. [2] *A unicyclic graph G is called cyclic caterpillar if the deletion of all its pendent vertices reduces it to a cycle.*

Definition 2.13. [2] *A thorn-regular cyclic caterpillar $C_{n,m}$ is a cyclic caterpillar with m vertices attached to each vertex $u_i, i = 1, ..., n$ of the parent cycle C_n.*

Let G be the cycle C_n on n vertices where $n \geq 3$ and H be a null graph N_m on m vertices, $m \geq 1$. Then the corona product $G \odot H$ will be a thorn-regular cyclic caterpillar $C_{n,m}$.

Now we state our first theorem as follows.

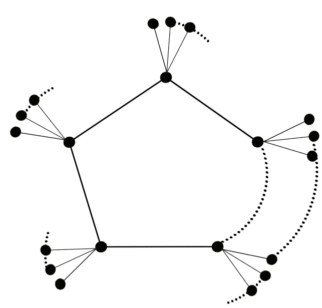

FIGURE 1. Corona product $C_n \odot N_m$

Theorem 2.2. *Let $G = C_n \odot N_m$ where $n \geq 3, m \geq 1$ be a thorn-regular cyclic caterpillar of order $n + nm$. Then for $v \in G$,*

$$l(v) = \begin{cases} \dfrac{m(m+1)}{(m+2)(m+3)} & \text{if } v \text{ is a node of } C_n \\ \dfrac{-(m+1)}{m+3} & \text{if } v \text{ is a pendant node} \end{cases}$$

Proof. Let the vertices of C_n be labeled as $\{u_1, u_2, ..., u_n\}$ and that of N_m be labeled as $\{v_1, v_2, ..., v_m\}$ where $n \geq 3$ and $m \geq 1$. In $C_n \odot N_m = C_{n,m}$, $N(u_i)= \{u_{i-1}, u_{i+1}, v_{i,k}\}$, $1 \leq k \leq m$. Hence $deg(u_i) = m + 2$ for $1 \leq i \leq n$. Also, $deg(v_{i,k}) = 1$ for $1 \leq k \leq m$ and $1 \leq i \leq n$. Hence for $u_1 \leq v \leq u_n$,

$$\begin{aligned} l(v) &= \frac{1}{m+2}\left(\frac{m(m+2-1)}{m+3}\right) \\ &= \frac{m(m+1)}{(m+2)(m+3)} \end{aligned}$$

Now for a pendant node $v_{i,k}$, $N(v_{i,k}) = \{u_i\}$ where $1 \leq k \leq m$ and $1 \leq i \leq n$. Hence if v is such a node,

$$\begin{aligned} l(v) &= \frac{1-(m+2)}{m+3} \\ &= \frac{-(m+1)}{m+3} \end{aligned}$$

□

2.1.3. *Corona product $N_m \odot C_n$.* In the next theorem we analyze the product $H \odot G$ where H is N_m and G is C_n.

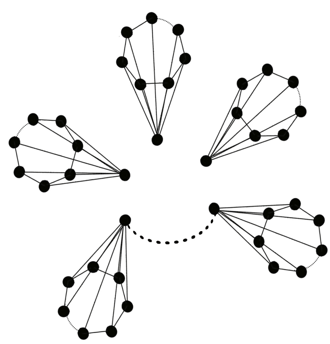

FIGURE 2. Corona product $N_m \odot C_n$

Theorem 2.3. *Let $G^* = N_m \odot C_n$. Then for $v \in G^*$,*

$$l(v) = \begin{cases} \dfrac{n-3}{n+3} & \text{if } v \text{ is a node of } N_m \\ \dfrac{-(n-3)}{3(n+3)} & \text{if } v \text{ is a node of } C_n. \end{cases}$$

Proof. Let the vertices of N_m be labeled as $\{v_1, v_2, ..., v_m\}$ and that of C_n be labeled as $\{u_1, u_2, ..., u_n\}$ where $n \geq 3$ and $m \geq 1$. In $N_m \odot C_n$, $N(v_i) = \{u_{i,k}\}$ where $1 \leq k \leq n$ and $1 \leq i \leq m$. Hence, $deg(v_i) = n$ for $1 \leq i \leq m$. Also, $deg(u_{i,k}) = 3$ for $1 \leq k \leq n$ and $1 \leq i \leq m$. Hence for $v_1 \leq v \leq v_m$,

$$\begin{aligned} l(v) &= \frac{1}{n}\left(\frac{n-3}{n+3}\right)n \\ &= \frac{n-3}{n+3} \end{aligned}$$

Finally, for a node $u_{i,k}$, $N(u_{i,k}) = \{v_i, u_{i,k-1}, u_{i,k+1}\}$ where $1 \leq k \leq n$ and $1 \leq i \leq m$. Hence if v is such a node,

$$\begin{aligned} l(v) &= \frac{1}{3}\left(\frac{3-n}{n+3}\right) \\ &= \frac{-(n-3)}{3(n+3)} \end{aligned}$$

□

Remark 2.3. *From the above analysis we see that in $C_n \odot N_m$, the leverage is completely depends on the number of vertices in N_m, and in $N_m \odot C_n$, it depends upon C_n. In both of the cases, there are two distinct leverage centralities even though C_n is null leverage connected and N_m is null leverage disconnected graph.*

For details on betweenness centrality in the context of the corona product, refer to [7].

2.2. **Leverage Centrality in Corona Product of Cycle with Complete graph.**

2.2.1. *Leverage Centrality in C_n and K_m.* As in the previous case, the leverage centrality of all the vertices of both of the component graphs C_n and K_m are zero since C_n is 2-regular and K_m is $m-1$ regular.

Now we present the detailed analysis of leverage centrality of vertices in the corona product of C_n and K_m in the following two theorems.

2.2.2. *Corona product $C_n \odot K_m$.*

Theorem 2.4. *Let $G^* = C_n \odot K_m$. Then for $v \in G^*$,*

$$l(v) = \begin{cases} \dfrac{m}{(m+1)(m+2)} & \text{if } v \text{ is a node of } C_n \\ \dfrac{-1}{m(m+1)} & \text{if } v \text{ is a node of } K_m \end{cases}$$

Proof. Let the vertices of C_n be labeled as $\{u_1, u_2, ..., u_n\}$ and that of K_m be labeled as $\{v_1, v_2, ..., v_m\}$ where $n \geq 3$ and $m \geq 2$. In $C_n \odot K_m$, $N(u_i)$= $\{u_{i-1}, u_{i+1}, v_{i,k}\}$, $1 \leq k \leq m$. So $deg(u_i) = m+2$ for $1 \leq i \leq n$. Now, $deg(v_{i,k}) = m$ for $1 \leq k \leq m$ and $1 \leq i \leq n$. Hence for $v \in C_n$

$$\begin{aligned} l(v) &= \frac{m}{m+2}\left(\frac{m+2-m}{2m+2}\right) \\ &= \frac{m}{(m+1)(m+2)} \end{aligned}$$

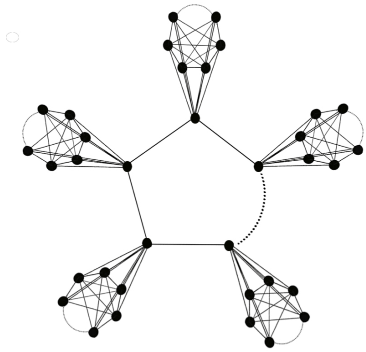

FIGURE 3. Corona product $C_n \odot K_m$

Now for a node $v_{i,k} \in K_m$, $N(v_{i,k}) = \{u_i, v_{i,j}\}$, $1 \leq j \leq m$, $j \neq k$ and $1 \leq i \leq n$. Also, $deg(v_{i,j}) = m$. Hence if v is such a node,

$$\begin{aligned} l(v) &= \frac{1}{m}\left(\frac{m-(m+2)}{2m+2}\right) \\ &= \frac{-1}{m(m+1)} \end{aligned}$$

□

2.2.3. *Corona product $K_m \odot C_n$.* Our next theorem is as follows.

Theorem 2.5. *Let $G^* = K_m \odot C_n$. Then for $v \in G^*$,*

$$l(v) = \begin{cases} \dfrac{n}{n+m-1}\left(\dfrac{n+m-4}{n+m+2}\right) & \text{if } v \text{ is a node of } K_m \\ \dfrac{-1}{3}\left(\dfrac{n+m-4}{n+m+2}\right) & \text{if } v \text{ is a node of } C_n \end{cases}$$

Proof. Let the vertices of K_m be labeled as $\{v_1, v_2, ..., v_m\}$ and that of C_n be labeled as $\{u_1, u_2, ..., u_n\}$ where $m \geq 2$ and $n \geq 3$. In $K_m \odot C_n$, $N(v_i) = \{u_{i,k}, v_j\}$ where $1 \leq j \leq m$, $j \neq i$ and $1 \leq k \leq n$. Hence $deg(v_i) = n+m-1$ for $1 \leq i \leq m$. Also, $deg(u_{i,k}) = 3$ for $1 \leq k \leq n$ and $1 \leq i \leq m$. Hence for $v_1 \leq v \leq v_m$,

$$\begin{aligned} l(v) &= \frac{n}{n+m-1}\left(\frac{n+m-1-3}{n+m+2}\right) \\ &= \frac{n}{n+m-1}\left(\frac{n+m-4}{n+m+2}\right) \end{aligned}$$

Now if $v \in C_n$, $v = u_{i,k}$ for $1 \leq i \leq m$ and $1 \leq k \leq n$. $N(u_{i,k}) = \{v_i, u_{i,k-1}, u_{i,k+1}\}$ where $1 \leq k \leq n$ and $1 \leq i \leq m$. Also we have $deg(v_i) = n+m-1$, $1 \leq i \leq m$. Hence if $v \in C_n$,

$$\begin{aligned} l(v) &= \frac{1}{3}\left(\frac{3-(n+m-1)}{n+m+2}\right) \\ &= \frac{-1}{3}\left(\frac{n+m-4}{n+m+2}\right) \end{aligned}$$

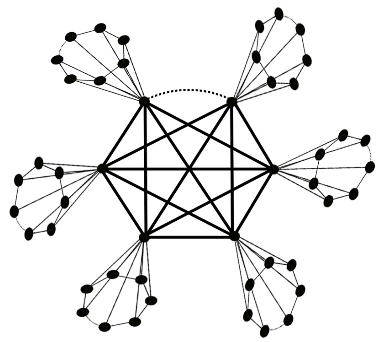

FIGURE 4. Corona product $K_m \odot C_n$

□

Remark 2.4. *From the above analysis we see that in $C_n \odot K_m$, the leverage is completely depends on the number of vertices in K_m, but in $K_m \odot C_n$, it depends upon both C_n and K_m. Here also in both of the cases, there are two distinct leverage centralities even though C_n and K_m are null leverage connected graphs.*

An analogous result for the classical betweenness centrality in the corona product of cycles with complete graphs is presented in [6]. From this, we observe that the betweenness centrality of any vertex in both $C_n \odot K_m$ and $K_m \odot C_n$ depends on both m and n, similar to the leverage centrality of nodes in $K_m \odot C_n$. However, the leverage centrality of any vertex in $C_n \odot K_m$ depends only on m.

2.3. **Leverage Centrality in Corona Product of Cycle with Path.**

2.3.1. *Leverage Centrality in C_n and P_m.* We know that $l(v) = 0\ \forall v \in C_n$. Now let the vertices of the path $P_m (m \geq 5)$ be labeled as $\{v_1, v_2, ..., v_m\}$. We have the following theorem.

Theorem 2.6. [1] *Let $G = P_m (m \geq 5)$ of order m. Then, for $v \in G$,*

$$l(v) = \begin{cases} \dfrac{-1}{3}, & \text{if } v = v_1, v_m \\ \dfrac{1}{6}, & \text{if } v = v_2, v_{m-1} \\ 0, & \text{if } v_3 \leq v \leq v_{m-2} \end{cases}$$

Thus there are three distinct leverage centralities in a path $P_m, (m \geq 5)$. Also, it is a bicentric leverage tree.

Now let us see the detailed analysis of leverage centrality of vertices in the corona product of C_n with P_m.

2.3.2. *Corona product $C_n \odot P_m$.* Let G be the cycle C_n on n vertices where $n \geq 3$ and H be a path P_m on m vertices, $m \geq 5$. The vertices of C_n be labeled as $\{u_1, u_2, ..., u_n\}$ and that of P_m be labeled as $\{v_1, v_2, ..., v_m\}$. Here the vertices in the copies H_i, $1 \leq i \leq n$ can be classified as:

- Type I: $deg(v) = 2$.

- Type II: $deg(v) = 3$ and is adjacent to the nodes of degree 2.
- Type III: $deg(v) = 3$ and is not adjacent to the nodes of degree 2.

We now proceed to state our next theorem.

Theorem 2.7. *Let $G^* = C_n \odot P_m$ where $n \geq 3$ and $m \geq 5$. Then for $v \in G^*$,*

$$l(v) = \begin{cases} \frac{1}{m+2}\left(\frac{m^3+3m^2+8}{(m+4)(m+5)}\right) & \text{if } v \text{ is a node of } C_n \\ \frac{-(3m+2)}{5(m+4)} & \text{if } v \text{ is a node of Type I} \\ \frac{2(5-2m)}{15(m+5)} & \text{if } v \text{ is a node of Type II} \\ \frac{-(m-1)}{3(m+5)} & \text{if } v \text{ is a node of Type III} \end{cases}$$

Proof. If v is a node of C_n, then $v = u_i$, $1 \leq i \leq n$. In the graph $C_n \odot P_m$, $N(u_i)$= $\{u_{i-1}, u_{i+1}, v_{i,k}\}$ where $1 \leq k \leq m$. So $deg(u_i) = m+2$ for $1 \leq i \leq n$. Here $deg(v_{i,k}) = 2$ for $k = 1, m$ and $deg(v_{i,k}) = 3$ for $2 \leq k \leq m-1$. Thus for $v \in C_n$,

$$\begin{aligned} l(v) &= \frac{1}{m+2}\left(\frac{2(m+2-2)}{m+4} + \frac{(m-2)(m+2-3)}{m+5}\right) \\ &= \frac{1}{m+2}\left(\frac{m^3+3m^2+8}{(m+4)(m+5)}\right) \end{aligned}$$

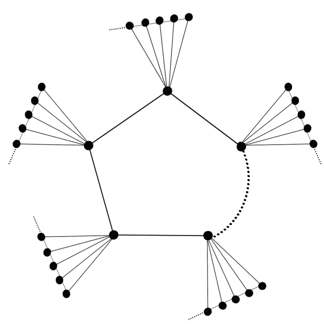

FIGURE 5. Corona product $C_n \odot P_m$

Now if v is a node of Type I, then $v = v_{i,k}$ for $k = 1, m$ and $1 \leq i \leq n$. In $C_n \odot P_m$, $N(v_{i,1})$= $\{u_i, v_{i,2}\}$ and $N(v_{i,m})$= $\{u_i, v_{i,m-1}\}$ where $1 \leq i \leq n$. Hence $deg(v_{i,k}) = 2$ for $k = 1, m$ and $1 \leq i \leq n$. Also, $deg(v_{i,2}) = deg(v_{i,m-1}) = 3$. Thus

$$\begin{aligned} l(v) &= \frac{1}{2}\left(\frac{2-3}{5} + \frac{2-(m+2)}{m+4}\right) \\ &= \frac{-(3m+2)}{5(m+4)} \end{aligned}$$

If v is a node of Type II, then $v = v_{i,k}$ for $k = 2, m-1$ and $1 \leq i \leq n$. In $C_n \odot P_m$, $N(v_{i,2})$= $\{u_i, v_{i,1}, v_{i,3}\}$ and $N(v_{i,m-1})$= $\{u_i, v_{i,m}, v_{i,m-2}\}$ where $1 \leq i \leq n$. Hence $deg(v_{i,k}) = 3$ for

$k = 2, m-1$ and $1 \le i \le n$. Now, $deg(v_{i,1}) = deg(v_{i,m}) = 2$ and $deg(v_{i,3}) = deg(v_{i,m-2}) = 3$. Thus

$$\begin{aligned} l(v) &= \frac{1}{3}\left(\frac{3-2}{5} + \frac{3-(m+2)}{m+5}\right) \\ &= \frac{-2(2m-5)}{15(m+5)} \end{aligned}$$

If v is a node of Type III, then $v = v_{i,k}$ for $3 \le k \le m-2$ and $1 \le i \le n$. In $C_n \odot P_m$, $N(v_{i,k})$= $\{u_i, v_{i,k-1}, v_{i,k+1}\}$ where $1 \le i \le n$. Hence, $deg(v_{i,k}) = 3$ for $3 \le k \le m-2$ and $1 \le i \le n$. Also, $deg(v_{i,k-1}) = deg(v_{i,k+1}) = 3$. Thus

$$\begin{aligned} l(v) &= \frac{1}{3}\left(\frac{3-(m+2)}{m+5}\right) \\ &= \frac{-(m-1)}{3(m+5)} \end{aligned}$$

□

2.3.3. *Corona product $P_m \odot C_n$.* Let the vertices of P_m be labeled as $\{v_1, v_2, ..., v_m\}$ and that of C_n be labeled as $\{u_1, u_2, ..., u_n\}$, where $m \ge 5$ and $n \ge 3$. In $P_m \odot C_n$, $deg(v_i) = n+1$ for $i = 1, m$ and $deg(v_i) = n+2$ for $2 \le i \le m-1$. Here the vertices in the path P_m, $m \ge 5$ can be classified as:

- Type I: $deg(v) = n+1$.
- Type II: $deg(v) = n+2$ and is adjacent to the nodes of degree $n+1$.
- Type III: $deg(v) = n+2$ and is not adjacent to the nodes of degree $n+1$.

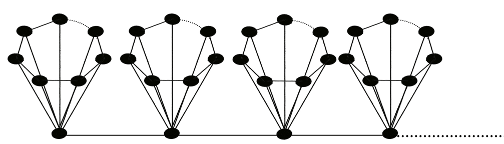

FIGURE 6. Corona product $P_m \odot C_n$

Finally, we state the following theorem.

Theorem 2.8. *Let $G^* = P_m \odot C_n$ where $m \ge 5$ and $n \ge 3$. Then for $v \in G^*$,*

$$l(v) = \begin{cases} \frac{1}{n+1}\left(\frac{-1}{2n+3} + \frac{n(n-2)}{n+4}\right) & \textit{if } v \textit{ is a node of Type I} \\ \frac{1}{n+2}\left(\frac{1}{2n+3} + \frac{n(n-1)}{n+5}\right) & \textit{if } v \textit{ is a node of Type II} \\ \frac{n(n-1)}{(n+2)(n+5)} & \textit{if } v \textit{ is a node of Type III} \\ \frac{-(n-2)}{3(n+4)} & \textit{if } v \in C_n \textit{ adjacent to Type I} \\ \frac{-(n-1)}{3(n+5)} & \textit{if } v \in C_n \textit{ adjacent to Type II or III} \end{cases}$$

Proof. If v is a node of Type I, then $v = v_i$ for $i = 1, m$. Here, $N(v_i)$= $\{u_{i,k}, v_{i+1}\}$ or $N(v_i)$= $\{u_{i,k}, v_{i-1}\}$ according as $i = 1$ or m where $1 \leq k \leq n$. Also, $deg(v_{i+1}) = deg(v_{i-1}) = n + 2$ and $deg(u_{i,k}) = 3$, $1 \leq k \leq n$. Hence

$$\begin{aligned} l(v) &= \frac{1}{n+1}\left(\frac{n(n+1-3)}{n+4} + \frac{(n+1)-(n+2)}{2n+3}\right) \\ &= \frac{1}{n+1}\left(\frac{-1}{2n+3} + \frac{n(n-2)}{n+4}\right) \end{aligned}$$

If v is a node of Type II, then $v = v_i$ for $i = 2, m-1$. Here, $N(v_i)$= $\{u_{i,k}, v_{i-1}, v_{i+1}\}$, $1 \leq k \leq n$. Now, $deg(v_{i-1}) = n + 1$, $deg(v_{i+1}) = n + 2$ for $i = 2$ and $deg(v_{i-1}) = n + 2$, $deg(v_{i+1}) = n + 1$ for $i = m - 1$. Also, $deg(u_{i,k}) = 3$, $1 \leq k \leq n$. Hence

$$\begin{aligned} l(v) &= \frac{1}{n+2}\left(\frac{n(n+2-3)}{n+5} + \frac{(n+2)-(n+1)}{2n+3}\right) \\ &= \frac{1}{n+2}\left(\frac{1}{2n+3} + \frac{n(n-1)}{n+5}\right) \end{aligned}$$

Now if v is a node of Type III, then $v = v_i$ for $3 \leq i \leq m-2$. Here, $N(v_i)$= $\{u_{i,k}, v_{i-1}, v_{i+1}\}$, $1 \leq k \leq n$. Also, $deg(v_{i-1})$= $deg(v_{i+1}) = n + 2$ and $deg(u_{i,k}) = 3$, $1 \leq k \leq n$. Hence

$$\begin{aligned} l(v) &= \frac{1}{n+2}\left(\frac{n(n+2-3)}{n+5}\right) \\ &= \frac{n(n-1)}{(n+2)(n+5)} \end{aligned}$$

Again if $v \in C_n$ is adjacent to Type I, then $v = u_{i,k}$ for $i = 1, m$ and $1 \leq k \leq n$. Here, $N(u_{i,k})$= $\{u_{i,k-1}, u_{i,k+1}, v_i\}$. Also, $deg(u_{i,k-1}) = deg(u_{i,k+1}) = 3$ and $deg(v_i) = n + 1$ for $i = 1, m$. Hence

$$\begin{aligned} l(v) &= \frac{1}{3}\left(\frac{3-(n+1)}{n+4}\right) \\ &= \frac{-(n-2)}{3(n+4)} \end{aligned}$$

Finally, if $v \in C_n$ is adjacent to Type II or Type III, then $v = u_{i,k}$ for $2 \leq i \leq m - 1$ and $1 \leq k \leq n$. Here, $N(u_{i,k})$= $\{u_{i,k-1}, u_{i,k+1}, v_i\}$. Also, $deg(u_{i,k-1}) = deg(u_{i,k+1}) = 3$ and $deg(v_i) = n + 2$ for $2 \leq i \leq m - 1$. Hence

$$\begin{aligned} l(v) &= \frac{1}{3}\left(\frac{3-(n+2)}{n+5}\right) \\ &= \frac{-(n-1)}{3(n+5)} \end{aligned}$$

□

Remark 2.5. *In $C_n \odot P_m$, $n \geq 3$ and $m \geq 5$ there are four distinct leverage centralities and all the vertices in C_n are leverage centers. But in $P_m \odot C_n$, $m \geq 5$ and $n \geq 3$ there are five distinct leverage centralities and which is a bicentric leverage graph. Type II nodes are leverage centers.*

In [9], an analogous result for the classical betweenness centrality in the corona product of cycles with path graphs is presented. For $C_n \odot P_m$ and $P_m \odot C_n$, the betweenness centrality depends on both m and n. In contrast, the leverage centrality of $C_n \odot P_m$ depends only on m, while that of $P_m \odot C_n$ depends only on n.

3. Numerical examples

In this section, we illustrate our results with some examples and practical applications to enhance the understanding of the concept. For theorem 2.2, let us take $G = C_5$ and $H = N_1$. Now, $C_5 \odot N_1$ is a thorn-regular cyclic caterpillar. For each node v of the cycle C_5, we now have

$$\begin{aligned} l(v) &= \frac{1}{3}\left(\frac{3-1}{3+1}\right) \\ &= \frac{1}{6} \end{aligned}$$

For the pendant node v,

$$\begin{aligned} l(v) &= \frac{1-3}{1+3} \\ &= \frac{-1}{2} \end{aligned}$$

If distinct numbers of vertices are added to each node of the cycle, the leverage centralities will vary, as the resulting graph is no longer a thorn-regular cyclic caterpillar, and the operation does not constitute a true corona product in this case. Therefore, we emphasize that there are only two distinct leverage centralities in the corona product $C_5 \odot N_1$.

For the corona product $N_m \odot C_n$, take $m = 1$ and $n = 3$. Then, the corona graph $N_1 \odot C_3$ becomes 3-regular, and hence the leverage centrality of all vertices is zero. This is consistent with our results in theorem 2.3. Similarly, for $C_n \odot K_m$, let us take $n = 3$ and $m = 2$. In $C_3 \odot K_2$, if v is a node of C_3, we have

$$\begin{aligned} l(v) &= \frac{1}{4}\left[2\left(\frac{4-4}{8}\right) + 2\left(\frac{4-2}{6}\right)\right] \\ &= \frac{1}{6} \end{aligned}$$

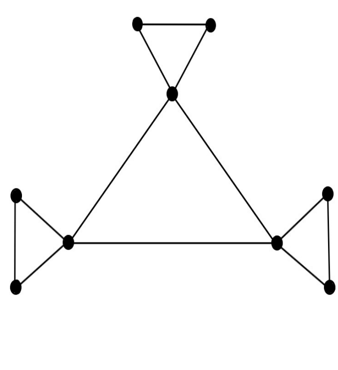

FIGURE 7. Corona product $C_3 \odot K_2$

Now, if v is a node of K_2, we have

$$l(v) = \frac{1}{2}\left[\left(\frac{2-2}{4}\right) + \left(\frac{2-4}{6}\right)\right] = \frac{-1}{6}$$

which agrees with our theorem 2.4. In $K_m \odot C_n$, consider $m = 2$ and $n = 3$ to get $K_2 \odot C_3$. If v is a node of K_2, we have

$$l(v) = \frac{1}{4}\left[3\left(\frac{4-3}{7}\right)\right] = \frac{3}{28}$$

If v is a node of C_3, we have

$$l(v) = \frac{1}{3}\left(\frac{3-4}{7}\right) = \frac{-1}{21}$$

The case of the corona product of a cycle with a path is similar.

3.1. **Practical Applications.**

3.1.1. *Cycle with Null Graph.* Consider a logistics network where C_n represents a central distribution ring connecting major warehouses, and N_m represents isolated storage units attached to each warehouse. Since the leverage centrality in $C_n \odot N_m$ is negative for the pendant nodes (storage units) and depends on the cycle nodes (warehouses), focus should be on the warehouses for network optimization. Improving infrastructure, security, and processing capabilities at the major warehouses (high-centrality nodes) will enhance the overall efficiency and robustness of the logistics network, ensuring smooth operations even if isolated storage units experience issues.

3.1.2. *Cycle with Complete Graph.* Imagine a communication network where C_n represents a ring of interconnected data centers, and K_m represents fully connected sub-networks of servers within each data center. The centrality of nodes in $K_m \odot C_n$ depends on both the size of the cycle n and the complete sub-graphs m, highlighting the critical nodes that ensure network integrity and performance. Enhancing bandwidth, redundancy, and security measures at the critical data centers (nodes with high leverage centrality) will significantly improve network resilience. This ensures that even if some servers or connections fail, the overall communication network remains robust and functional.

3.1.3. *Cycle with Path Graph.* Consider a public transportation network where C_n represents a circular metro line connecting key city areas, and P_m represents linear bus routes extending into suburban regions from each metro station. In $C_n \odot P_m$, the leverage centrality of nodes in the metro line (cycle) is influenced by the length of the bus routes (paths), emphasizing the importance of these connecting nodes. Investing in metro stations with high leverage centrality by improving facilities, accessibility, and connectivity will enhance the overall efficiency and user experience. This ensures that key transportation hubs effectively support suburban commutes and reduce congestion.

4. Conclusions

By comparing leverage centrality across different graph structures, we can better understand how changes in graph composition affect centrality measures. Null graph N_m and the complete graph K_m are null leverage graphs in two different ways, but the corona product with the connected null leverage cycle C_n gives interesting results. Also, we observe that the betweenness centrality of any vertex in $C_n \odot K_m$, $K_m \odot C_n$, $C_n \odot P_m$ and $P_m \odot C_n$ depends on both m and n, similar to the leverage centrality of nodes in $K_m \odot C_n$. However, the leverage centrality of any vertex in $C_n \odot K_m$ and $C_n \odot P_m$ depends only on m. P_m is a bicentric leverage tree and the leverage type of P_m is preserved in the product $P_m \odot C_n$, $m \geq 5$ and $n \geq 3$, but which is not a tree. This study can be extended to other graph products.

Acknowledgments

We thank the referees for their time and comments.

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