CREAT. MATH. INFORM.
Volume **34** (2025), No. 1,
Pages 87-95

Online version at https://creative-mathematics.cunbm.utcluj.ro/
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2025.01.09

Some new results on WS-algebras

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ABSTRACT. The concept of WS-algebras was introduced in 2008 by K. J. Lee, J. B. Young and H. Kim where some important basic properties of this class of algebras are given. In this paper, some other important properties of WS-algebras are registered and, in addition, some types of ideals in such algebras are discussed. Also, some information about atoms in this class of algebra is offered.

1. INTRODUCTION

The concept of subtraction algebras appeared in 1995 in [8], written by Zelinka. The concept of weak subtraction algebras appeared in 2008 in [4] written by K. J. Lee, J. B. Young and H. Kim. This class of logical algebras has rarely been the subject of research interest since then. There is a well-known paper [2] written by Y. B. Jun, C. H. Park and E. H. Roh in which ideals in WS-algebras were discussed, as well as a paper [6] in which connections of this class of algebras with some other known algebras were treated. In articles [1] and [5], the subject of research is soft WS-algebras.

In this article, besides adding some new properties to this class of logical algebras, discussing ideals, we introduce and analyze p-ideals and implicative ideals in WS-algebras. Also, discussing the extension of the WS-algebra to the WS-algebra by adding one element, we register some properties of the atoms in these algebras. The paper is designed as follows: In the Preliminaries section, we give the necessary terms and their interrelationships so that the material presented in Section 3, the main part of this report, would be readable for the reader. Section 3 has three subsections. In the first of them, some additional properties about WS-algebras that are not present in the available sources are registered. The second subsection discusses the concept of ideals in this class of algebras. Thus, some newly observed properties of ideals are registered (Theorem 3.4). In addition, two new classes of ideals are recognized: p-ideals (Definition 3.1) and implicative ideals (Definition 3.2). In subsection 3.3, an extension of a WS-algebra $\mathfrak{A} = (A, \cdot, 1)$ to the WS-algebra $\mathfrak{B} = (A \cup \{a\}, *, 1)$ is designed by adding an element (Theorem 3.7) so that the newly created algebra has an atom in it. Further on, some properties of atoms in WS-algebras are studied. Thus, some properties of atoms in WS-algebras have been identified and a criterion for recognizing atoms in such algebras has been found.

2. PRELIMINARIES

The concept of subtraction algebras appears in 1995 in [8], written by Zelinka. An algebra $(A, \cdot, 1)$ is a subtraction algebra (shortly, S-algebra) if it satisfies the following conditions

(Re) $(\forall x \in A)(x \cdot x = 1)$,

(Im) $(\forall z, y \in A)(x \cdot (y \cdot x) = x)$,

Received: 11.03.2024. In revised form: 03.08.2024. Accepted: 31.08.2024
2020 *Mathematics Subject Classification.* 03G25, 06D20, 06F35.
Key words and phrases. *S-algebra, WS-algebra, ideal, p-ideal, implicative ideal, atom.*

(S2) $(\forall x, y \in A)(x \cdot (x \cdot y) = y \cdot (y \cdot x))$,

(Ex) $(\forall x, y, z \in A)((x \cdot y) \cdot z = (x \cdot z) \cdot y)$.

We note that condition (S2), in this terminology, is recognized as the commutativity condition of logical algebras. In [7] it was shown that condition (Re) can be demonstrated from the remaining conditions. Also, it was shown that:

- an S-algebra is equivalent to an implicative BCK-algebra. ([3], Theorem 2)
- a dual S-algebra (in the standard sense, $x * y =: y \cdot x$ for any $x, y \in A$) is a commutative self-distributive BE-algebra. ([6], Theorem 4.4)
- a dual S-algebra is a commutative Hilbert algebra. ([6], Corollary 4.5)

The concept of weak subtraction algebras (shortly, WS-algebras) appeared in 2008 in [4] written by K. J. Lee, J. B. Young and H. Kim. By a weak subtraction algebra (shortly, WS-algebra), we mean a triplet $\mathfrak{A} =: (A, \cdot, 1)$ which satisfies the following axioms:

(Re) $(\forall x \in A)(x \cdot x = 1)$,

(M) $(\forall x \in A)(x \cdot 1 = x)$,

(Ex) $(\forall x, y, z \in A)((x \cdot y) \cdot z = (x \cdot z) \cdot y)$,

(DR) $(\forall x, y, z \in A)((x \cdot y) \cdot z = (x \cdot z) \cdot (y \cdot z))$.

We denote this axiomatic system by **WS** and the corresponding algebraic structure, determined by it, by WS-algebra. The algebra $(\{1\}, \cdot, 1)$ is a trivial WS-algebra. Of course, as it was shown in [4], Theorem 3.6, every S-algebra is a WS-algebra, but the reverse need not be the case ([6], Example 2.9). In addition, it was shown in [6] that:

- every commutative self-distributive BE-algebra is a dual WS-algebra.
- every Hilbert algebra is a dual WS-algebra.
- every Heyting algebra is a dual WS-algebra.

Some important properties of WS-algebra are given by the following proposition:

Proposition 2.1 ([4], Theorem 3.3). *Let $\mathfrak{A} =: (A, \cdot, 1)$ be a WS-algebra. Then:*

(L) $(\forall x \in A)(1 \cdot x = 1)$

(2) $(\forall x, y \in A)((x \cdot y) \cdot x = 1)$

(3) $(\forall x, y \in A)(x \cdot y = 1 \implies (x \cdot z) \cdot (y \cdot z) = 1)$.

Example 2.1. *Let $A = \{1, a, b, c, d\}$ and operation $'\cdot'$ defined on A as follows:*

$\cdot$	1	a	b	c	d
1	1	1	1	1	1
a	a	1	1	1	1
b	b	b	1	b	b
c	c	c	c	1	c
d	d	d	d	d	1

Then $\mathfrak{A} = (A, \cdot, 1)$ is a WS-algebra ([2], *Example 4.2(3)*) *but it is not a subtraction algebra because, for example, $a \cdot (b \cdot a) = a \cdot b = 1 \neq a$ holds.*

3. The main results

3.1. **Some basic properties of WS-algebras.** If we define the relation $\preccurlyeq$ on a WS-algebra $\mathfrak{A} =: (A, ., 1)$ as follows

$$(\forall x, y \in A)(x \preccurlyeq y \iff y \cdot x = 1),$$

it was shown ([4], Proposition 3.8) that this relation is a quasi-order relation on A (reflexive and transitive relation) compatible with the operation in $\mathfrak{A}$. Due to the consistency of the material presented in this report, we present the proof of this claim here:

Proposition 3.2. *The relation* $\preccurlyeq$, *determined as above, on a WS-algebra* $\mathfrak{A} =: (A, \cdot, 1)$ *is a quasi-ordered on* A *comparable with the operation in* $\mathfrak{A}$ *in the following sense:*

(4) $(\forall x, y, z \in A)(x \preccurlyeq y \implies x \cdot z \preccurlyeq y \cdot z)$,

(5) $(\forall x, y, z \in A)(x \preccurlyeq y \implies z \cdot x \preccurlyeq z \cdot y)$.

Proof. The relation $\preccurlyeq$ is reflexive due to (Re).

To prove transitivity, suppose that $x, y, z \in A$ are arbitrary elements such that $x \preccurlyeq y$ and $y \preccurlyeq z$. This means $y \cdot x = 1$ and $z \cdot y = 1$. If we put $x = z$ and $z = x$ in (3), we get

$$1 = z \cdot y \implies 1 = (z \cdot x) \cdot (y \cdot x) = (z \cdot x)1 = z \cdot x.$$

Thus $z \cdot x = 1$ which means $x \preccurlyeq z$.

Let $x, y, z \in A$ be arbitrary elements such that $x \preccurlyeq y$. This means $y \cdot x = 1$. Thus $(y \cdot z) \cdot (x \cdot z) = 1$ according to (3). Hence, $x \cdot z \preccurlyeq y \cdot z$.

Let $x, y, z \in A$ be arbitrary elements such that $x \preccurlyeq y$. This means $y \cdot x = 1$. Then $(y \cdot x) \cdot z = 1 \cdot z = 1$ by (L). On the other hand, by (DR), we have $(y \cdot z) \cdot (x \cdot z) = 1$. So, $x \cdot z \preccurlyeq y \cdot z$. □

Example 3.2. *Let* $A = \{1, a, b, c, d\}$ *as in Example 2.1. The relation* $\preccurlyeq$ *is given by* $\preccurlyeq = \{(1, 1), (a, 1), (b, 1), (c, 1), (d, 1), (a, a), (b, a), (c, a), (d, a), (b, b), (c, c), (d, d)\}$.

Note here that, analogously to Proposition 3.4 in [4], it can be shown that if a WS-algebra $\mathfrak{A} = (A, \cdot, 1)$ satisfies the additional condition

($\star$) $(\forall x, y \in A)(x \cdot (x \cdot y) = y \cdot (y \cdot x))$,

then $\preccurlyeq$ is an antisymmetric relation. An example of such a WS-algebra satisfying the condition ($\star$) is the algebra in Example 2.1 However, in the general case, a WS-algebra cannot satisfy the condition ($\star$), as the following example shows.

Example 3.3. *Let* $A = \{1, a, b, c\}$ *and operation '$\cdot$' defined on* A *as follows:*

$\cdot$	1	a	b	c
1	1	1	1	1
a	a	1	a	a
b	b	b	1	1
c	c	c	1	1

Then $\mathfrak{A} = (A, \cdot, 1)$ *is a WS-algebra* ([4], *Example 3.2*) *but which does not satisfy the condition ($\star$) because, for example, we have* $b \cdot (b \cdot c) = b \cdot 1 = b \neq c = c \cdot 1 = c \cdot (c \cdot b)$.

In addition to the above, this relation also has the following characteristics:

Proposition 3.3. *Let* $\mathfrak{A} = (A, \cdot, 1)$ *be a WS-algebra. Then:*

(6) $(\forall x \in A)(x \preccurlyeq 1)$,

(7) $(\forall x, y, z \in A)(z \preccurlyeq x \cdot y \iff y \preccurlyeq x \cdot z)$,

(8) $(\forall x, y \in A)(x \preccurlyeq x \cdot y)$,

(9) $(\forall x, y \in a)(y \preccurlyeq x \cdot (x \cdot y))$.

(10) $(\forall x, y, z \in A)(x \cdot y \preccurlyeq (x \cdot z) \cdot (y \cdot z))$.

Proof. The validity of (6) follows from the validity of formula (L).

(7) is obtained directly from (Ex).

(8) can be obtained from (2).

(9) Since for arbitrary $x, y \in A$ we have $(x \cdot y) \cdot (x \cdot y) = 1$ according to (Re), from here, in accordance with (Ex), we get $(x \cdot (x \cdot y)) \cdot y = 1$ which gives (9).

(10) For arbitrary $x, y, z \in A$, according to (8), we have $x \preccurlyeq x \cdot z \preccurlyeq (x \cdot z) \cdot (y \cdot z)$. □

In what follows, we base ourselves on the following statements: Let $\mathfrak{A} = (A, \cdot, 1)$ be a WS-algebra. A quasi-order relation $\preccurlyeq$ on a set A generates the equivalence relation $\equiv_{\preccurlyeq} =: \preccurlyeq \cap \preccurlyeq^{-1}$ on A. Due to properties (4) and (5), this equivalence is a congruence on $\mathfrak{A}$. According to what was shown in Theorem 3.10 in article [4], for any congruence q on the WS-algebra A one can design a quotient WS-algebra $\mathfrak{A}/q = (A/q, *[1]_q)$. So, $\mathfrak{A}/\equiv_{\preccurlyeq}$ is a WS-algebra.

Also, we have:

Theorem 3.1. *Let $A = (A, \cdot, 1)$ a WS-algebra. Then:*

(a) $(\forall x, y \in A)((x \cdot y) \cdot y = x \cdot y)$

(b) $(\forall x, z \in A)((x \cdot z) \cdot (y \cdot z) = (x \cdot y) \cdot (z \cdot y))$

Proof. (a) If we put $z = y$ in (DR), we get

$$(x \cdot y) \cdot y = (x \cdot y) \cdot (y \cdot y) = (x \cdot y) \cdot 1 = x \cdot y$$

according to (Re) and (M).

(b) If (Ex) and (DR) are combined, we get (b) □

In what follows, the following lemma is important:

Lemma 3.1. *Let $\mathfrak{A} =: (A, \cdot, 1)$ be a WS-algebra. Then, $\mathfrak{A}$ satisfies the condition*

$$(\forall x, y, z \in A)(x \cdot (y \cdot z) = y \cdot (x \cdot z))$$

if and only if $\mathfrak{A}$ satisfies the condition $(\forall x, y \in A)(x \cdot y = y \cdot x)$.

Proof. Suppose that a WS-algebra $\mathfrak{A}$ satisfies the condition $x \cdot (y \cdot z) = y \cdot (x \cdot z)$ for arbitrary elements $x, y, z \in A$. If we choose $z = 1$, from this formula we get $x \cdot y = y \cdot x$ with respect to (M). Conversely, suppose that $x \cdot y = y \cdot x$ is valid for arbitrary $x, y, z \in A$. We have $x \cdot (y \cdot z) = x \cdot (z \cdot y) = (z \cdot y) \cdot x = (z \cdot x) \cdot y = y \cdot (z \cdot x) = y \cdot (x \cdot z)$. □

However, we have:

Theorem 3.2. *A non-trivial WS-algebra $\mathfrak{A} =: (A, \cdot, 1)$ cannot satisfy the condition*

$$(\forall x, y, z \in A)(x \cdot (y \cdot z) = y \cdot (x \cdot z)).$$

Proof. Assume that the non-trivial WS-algebra $\mathfrak{A} =: (A, \cdot, 1)$ satisfies the mentioned condition. Then, for arbitrary $x, y, z \in A$, we would have $(x \cdot y) \cdot z = z \cdot (x \cdot y) = x \cdot (z \cdot y) = (z \cdot y) \cdot x = (z \cdot x) \cdot y$. From here, for $x = y = 1$, we would get $1 = 1 \cdot z = (1 \cdot 1) \cdot z = (z \cdot 1) \cdot 1 = z \cdot 1 = z$, that is, with respect to (Re) , (L) and (M), we would get $1 = 1 \cdot z = z$ which is contradictory to (L). The resulting contradiction refutes the assumption. □

3.2. **On ideals in WS-algebras.** The concept of ideals in WS-algebras is introduced in [2], Definition 3.1, as follows:

Let $\mathfrak{A} = (A, \cdot, 1)$ be a WS-algebra. A non-empty subset J of A is an ideal in $\mathfrak{A}$ if it satisfies:

(J1) $1 \in J$,

(J2) $(\forall z, y \in A)((x \cdot y \in J \wedge y \in J) \implies x \in J)$.

Theorem 3.3. *Let $\mathfrak{A} = (A, \cdot, 1)$ be a WS-algebra and let J be a subset of A satisfying the condition (J1). Then the condition (J2) is equivalent to the condition*

(J3) $(\forall x, y, z \in A)(((x \cdot y) \cdot z \in J \wedge y \in J) \implies x \cdot z \in J)$.

Proof. If we put $x = x \cdot z$ in (J2), and replace $(x \cdot z) \cdot y \in J$ with $(x \cdot y) \cdot z \in J$ (which is allowed because of (Ex)), we get (J3).

The validity of the formula (J2) is obtained from the validity of the formula (J3) if we choose $z = 1$ with respect to (M). □

Theorem 3.4. *For an ideal J in a WS-algebra $\mathfrak{A} = (A, \cdot, 1)$ holds:*

(J4) $(\forall x, y \in A)((x \in J \wedge x \preccurlyeq y) \implies y \in J)$.

(J5) $(\forall y, z \in A)(y \in J \implies y \cdot z \in J)$.

(J6) $(\forall x, y \in A)((x \notin J \wedge y \in J) \implies x \cdot y \notin J)$.

Proof. Let $x, y \in A$ be arbitrary elements such that $x \in J$ and $x \preccurlyeq y$. This means $x \in J$ and $y \cdot x = 1 \in J$. Thus $y \in J$ by (J2).

Let J be an ideal in a WS-algebra $\mathfrak{A}$. So, (J3) is a valid formula in $\mathfrak{A}$. If we put $x = y$ in (J3), we get (J5) using (Re) and (L).

Let $x, y \in J$ be such that $x \notin J$ and $y \in J$. Since the contraposition of (J2) is $x \notin J \implies (y \notin J \vee x \cdot y \notin J)$, we conclude that it must be $x \cdot y \notin J$ because $y \in J$ is an assumption. □

Example 3.4. *Let $A = \{1, a, b, c, d\}$ as in Example 2.1. Subsets $J_0 = \{1\}$, $J_1 = \{1, a\}$, $J_2 = \{1, a, b\}$, $J_3 = \{1, a, c\}$, $J_4 = \{1, a, d\}$, $J_5 = \{1, a, b, c\}$, $J_6 = \{1, a, b, d\}$ and $J_7 = \{1, a, c, d\}$ are ideals in $\mathfrak{A}$. However, the subsets $B = \{1, b, c\}$, $C = \{1, b, d\}$, $D = \{1, c, d\}$ and $E = \{1, b, c, d\}$ are not ideals in $\mathfrak{A}$.*

Let us recall that the concept of subalgebra in WS-algebras is introduced in a standard way: A nonempty subset S of a WS-algebra A is a subalgebra in A if and only if

(S2) $(\forall x, y \in A)((x \in S \wedge y \in S) \implies x \cdot y \in S)$ holds.

It can be easily concluded that any subalgebra in a WS-algebra $\mathfrak{A}$ satisfies the condition

(S1) $1 \in S$.

Assertion (J5) of the previous theorem allows us to state an important consequence about ideals in WS-algebras.

Corollary 3.1. *Every ideal in a WS-algebra $\mathfrak{A}$ is a subalgebra in $\mathfrak{A}$.*

The converse statement need not be valid as the following example shows.

Example 3.5. *Let $A = \{1, a, b, c, d\}$ as in Example 2.1. The subset $B = \{1, b, c\}$ in a WS-algebra $\mathfrak{A} = (A, \cdot, 1)$ is a subalgebra in $\mathfrak{A}$ but it is not an ideal in $\mathfrak{A}$ (see Example 3.4)*

The concept of p-ideals in WS-algebras is introduced in the standard way:

Definition 3.1. *Let $A = (A, \cdot, 1)$ be a WS-algebra. A non-empty subset J of A is a p-ideal in $\mathfrak{A}$ if the following holds:*

(J1) $1 \in J$,

(J7) $(\forall x, y, z \in A)(((x \cdot z) \cdot (y \cdot z) \in J \wedge y \in J) \implies x \in J)$.

Example 3.6. *Let $A = \{1, a, b, c, d\}$ as in Example 2.1. Then $\mathfrak{A} = (A, \cdot, 1)$ is a WS-algebra. Ideal $J_2 = \{1, a, b\}$ is a p-ideal in $\mathfrak{A}$.*

Theorem 3.5. *Let $A = (A, \cdot, 1)$ be a WS-algebra. Any p-ideal in the WS-algebra $\mathfrak{A}$ is an ideal in $\mathfrak{A}$.*

Proof. If we put $z = 1$ in (J7), we get (J2). □

However, every deal in a WS-algebra $\mathfrak{A}$ need not be a p-ideal in $\mathfrak{A}$, as the following example shows:

Example 3.7. *Let $A = \{1, a, b, c, d\}$ as in Example 2.1. Then $\mathfrak{A} = (A, \cdot, 1)$ is a WS-algebra. The ideal $J_3 = \{1, a, c\}$ in $\mathfrak{A}$ is not a p-ideal in $\mathfrak{A}$ because, for example, we have $(d \cdot c) \cdot (a \cdot c) = d \cdot 1 = 1 \in J$ and $a \in J$ but $d \notin J$.*

We also introduce the concept of implicative ideals in a WS-algebra in the standard way:

Definition 3.2. *Let $\mathfrak{A} = (A, \cdot, 1)$ be a WS-algebra. A non-empty subset J of A is an implicative ideal in $\mathfrak{A}$ if the following holds:*

(J1) $1 \in J$,

(J8) $(\forall x, y, z \in A)((x \cdot y) \cdot z \in J \wedge y \cdot z \in J) \implies x \cdot z \in J)$.

Proposition 3.4. *Let J be an implicative ideal in a WS-algebra $\mathfrak{A}$. Then*

(J9) $(\forall x, y \in A)((x \cdot y) \cdot y \in J \implies x \cdot y \in J)$.

Proof. If we put $z = y$ in (J8), we obtain (J9) with respect to (Re) and (J1). □

Theorem 3.6. *Any implicative ideal in a WS-algebra $\mathfrak{A}$ is an ideal in $\mathfrak{A}$.*

Proof. Let J be an implicative ideal in a WS-algebra $\mathfrak{A}$ and let $x, y \in A$ be such that $x \cdot y \in J$ and $y \in J$. Then $(x \cdot y) \cdot 1 = x \cdot y \in J$ and $y \cdot 1 = y \in J$ in accordance with (M). Thus $x = x \cdot 1 \in J$ by (J9). So, J is an ideal in $\mathfrak{A}$. □

Example 3.8. *Let $A = \{1, a, b, c, d\}$ as in Example 2.1. Then $\mathfrak{A} = (A, \cdot, 1)$ is a WS-algebra. The ideal $J_4 = \{1, a, d\}$ is an implicative ideal in $\mathfrak{A}$.*

However, every ideal in a WS-algebra $\mathfrak{A}$ need not be an implicative ideal in $\mathfrak{A}$ as the following example shows:

Example 3.9. *Let $A = \{1, a, b, c, d\}$ as in Example 2.1. Then $\mathfrak{A} = (A, \cdot, 1)$ is a WS-algebra. The ideal $J_0 = \{1\}$ is not an implicative ideal in $\mathfrak{A}$ because, for example, we have $(b \cdot a) \cdot a = a \cdot a = 1 \in J_0$ but $b \cdot a = a \notin J_0$.*

3.3. **Extension of a WS-algebra to the WS-algebra by adding one element.** In [4], Theorem 3.13, a procedure for constructing a WS-algebra based on a given quasi-ordered set was created.

Proposition 3.5. *Let $(X, \preccurlyeq)$ be a quasi-ordered set and $a \notin X$. If we put $Y = X \cup \{a\}$ and define the operation $*$ on Y as follows $x * y =: a$ if $x \preccurlyeq y$ and $x * y =: x$ in other cases, then $(Y, *, a)$ is a WS-algebra.*

The following example illustrates the previous proposition where the elements of the set $A = \{1, a, b, c\}$ are not comparable. So, the order relation $\preccurlyeq$ is given by

$$\preccurlyeq =: \{(1, 1), (a, 1), (a, a), (b, 1), (b, b), (c, 1), (c, c)\}.$$

Example 3.10. *Let $A = \{1, a, b, c\}$ and operation $'\cdot'$ defined on A as follows:*

$\cdot$	1	a	b	c
1	1	1	1	1
a	a	1	a	a
b	b	b	1	b
c	c	c	c	1

Then $\mathfrak{A} = (A, \cdot, 1)$ is a WS-algebra.

In the following theorem, we give a construction extending the WS-algebra to the WS-algebra by adding one element:

Theorem 3.7. *Let* $(A, \cdot, 1)$ *be a WS-algebra and let* $a \notin A$. *If we define the operation* $*$ *on the set* $B =: A \cup \{a\}$ *as follows*

$$x * y = \begin{cases} x \cdot y & \text{for } x \in A \ \wedge \ y \in A, \\ a & \text{for } x = a \ \wedge \ y = 1, \\ 1 & \text{for } x = 1 \ \wedge \ y = a, \\ 1 & \text{for } x = a \ \wedge \ y = a, \\ a & \text{for } x = a \ \wedge \ y \in A \setminus \{1\}, \\ x & \text{for } x \in A \setminus \{1\} \ \wedge \ y = a, \end{cases}$$

then $(A \cup \{a\}, *, 1)$ *is a WS-algebra.*

Proof. The proof of this theorem can be obtained by direct checking, putting the element a once, twice or three times instead of the variables x, y and z in formulas (Ex) and (DR). □

One specificity of the WS-algebra created in this way should be noted. Namely, the element a has the following property

$$(\forall x \in A)(a \preccurlyeq x \implies (x = a \vee x = 1)).$$

We recognize this property in any logical algebra $\mathfrak{A} = (A, \cdot, 1)$ as an 'atom in the algebra $\mathfrak{A}$'. We introduce such a determination in the present case, the WS-algebra case: For an element a of the WS-algebra $\mathfrak{A} = (A, \cdot, 1)$ we say that an atom is in $\mathfrak{A}$ if it satisfies the condition

(At) $(\forall x \in A)(a \preccurlyeq x \implies (x \equiv_{\preccurlyeq} a \vee x \equiv_{\preccurlyeq} 1))$.

We denote the set of all atoms in the WS-algebra $\mathfrak{A}$ by $L(A)$.

Example 3.11. *(1) Let* $A = \{1, a, b, c, d\}$ *be as in Example 2.1. Then* $\mathfrak{A} = (A, \cdot, 1)$ *is a WS-algebra. The element* a *is an atom in* $\mathfrak{A}$.

(2) Let $A = \{1, a, b, c\}$ *be as in Example 3.3. Then* $\mathfrak{A} = (A, \cdot, 1)$ *is a WS-algebra. The elements* a, b *are atoms in* $\mathfrak{A}$.

(3) Let $A = \{1, a, b, c\}$ *be as in Example 3.10. Then* $\mathfrak{A} = (A, \cdot, 1)$ *is a WS-algebra. In this case, all elements, except* 1, *are atoms of the WS-algebra* $\mathfrak{A}$. *Therefore,* $L(A) = A \setminus \{1\}$.

Proposition 3.6. *If for the element* a *of the WS-algebra* $\mathfrak{A} = (A, \cdot, 1)$, *the subset* $\{1, a\}$ *is an ideal in* $\mathfrak{A}$, *then* a *is an atom in* $\mathfrak{A}$.

Proof. Let the subset $\{1, a\}$ be an ideal in a WS-algebra $\mathfrak{A}$, and let $a \preccurlyeq x$ be valid. Then $x \cdot a = 1 \in \{1, a\}$. Thus $x = 1 \vee x = a$ since $\{1, a\}$ is an ideal in $\mathfrak{A}$. So, the element a is an atom in $\mathfrak{A}$. □

Also, the following applies:

Proposition 3.7. *Let* a *be an atom in a WS-algebra* $\mathfrak{A} = (A, \cdot, 1)$. *Then following holds:*

(11) $(\forall y \in A)(a \cdot y \equiv_{\preccurlyeq} 1 \vee a \cdot y \equiv_{\preccurlyeq} a)$,

(12) $(\forall x \in A)(a \equiv_{\preccurlyeq} x \cdot (x \cdot a) \vee x \equiv_{\preccurlyeq} x \cdot a)$.

Proof. If we put $x = a$ in the valid formula (8), we get $a \preccurlyeq a \cdot y$. Since a is an atom in $\mathfrak{A}$, it must be $a \cdot y \equiv_{\preccurlyeq} 1$ or $a \cdot y \equiv_{\preccurlyeq} a$.

If we put $y = a$ in the valid formula (9), we get $a \preccurlyeq x \cdot (x \cdot a)$. Since a is an atom in $\mathfrak{A}$, it must be $x \cdot (x \cdot a) \equiv_{\preccurlyeq} 1$ or $x \cdot (x \cdot a) \equiv_{\preccurlyeq} a$. Assume that $x \cdot (x \cdot a) \equiv_{\preccurlyeq} 1$ holds. This means $x \cdot a \preccurlyeq x$. Then $x \cdot a \preccurlyeq x$ and $x \preccurlyeq x \cdot a$ by (8). Thus $x \equiv_{\preccurlyeq} x \cdot a$. □

The converse of statement (12) also holds:

Theorem 3.8. *If an element a of a WS-algebra $\mathfrak{A} = (A, \cdot, 1)$ satisfies the condition (12), then a is an atom in $\mathfrak{A}$.*

Proof. Let the element a of a WS-algebra $A = (A, \cdot, 1)$ satisfies the condition (12) and let $x \in A$ be such that $a \preccurlyeq x$. If $x \cdot (x \cdot a) \equiv_{\preccurlyeq} a$ holds, then we have $x \cdot (x \cdot a) \preccurlyeq x$. On the other hand, according to (8), we have $x \preccurlyeq x \cdot (x \cdot a)$. Therefore, $x \equiv_{\preccurlyeq} x \cdot (x \cdot a) \equiv_{\preccurlyeq} a$. If $x \cdot (x \cdot a) \equiv_{\preccurlyeq} 1$ holds, then we have $xa \preccurlyeq x$. On the other hand, according to (8), we have $x \preccurlyeq x \cdot a$. Therefore, $x \equiv_{\preccurlyeq} x \cdot a = 1$. □

Let us state some properties of the set $L(A)$ of all atoms of the WS-algebra $\mathfrak{A} = (A, \cdot, 1)$.

Theorem 3.9. *Let $\mathfrak{A} = (A, \cdot, 1)$ be a WS-algebra. Then:*

(13) *$L(A)$ is an anti-chain.*

(14) *$(L(A) \cup \{1\})/ \equiv_{\preccurlyeq}$ is a sub-algebra in $\mathfrak{A}/ \equiv_{\preccurlyeq}$.*

Proof. Let $a, b\,(\neq a)$ be atoms in a WS-algebra $\mathfrak{A}$. If there were $a \preccurlyeq b$, we would have $a \equiv_{\preccurlyeq} b$ or $b \equiv_{\preccurlyeq} 1$, which is impossible.

If $a, b \in A$ are atoms in a WS-algebra $\mathfrak{A}$, then, according to (11), we have $a \cdot b \equiv_{\preccurlyeq} 1$ or $a \cdot b \equiv_{\preccurlyeq} a$. How the first option, according to (13), it is not possible, it must be $a \cdot b \equiv_{\preccurlyeq} a$ which means that $(L(A) \cup \{1\})/ \equiv_{\preccurlyeq}$ is a sub-algebra in $\mathfrak{A}/ \equiv_{\preccurlyeq}$. □

We end this subsection with a statement about the WS-algebra $\mathfrak{A} = (A, \cdot, 1)$ whose elements, except 1, are atoms of $\mathfrak{A}$.

Theorem 3.10. *Let $\mathfrak{A} = (A, \cdot, 1)$ be a WS-algebra whose elements, except 1, are atoms in $\mathfrak{A}$. Then every sub-algebra in $\mathfrak{A}$ is an ideal in $\mathfrak{A}$.*

Proof. Let S be a sub-algebra in a WS-algebra $\mathfrak{A}$ and let $x, y \in A$ be such that $x \cdot y \in S$ and $y \in S$. Since x, y are atoms in A, according to (11), we have $x \cdot y \equiv_{\preccurlyeq} x$ because the option $x \cdot y \equiv_{\preccurlyeq} 1$ is not possible according to (13). Therefore, $x \equiv_{\preccurlyeq} x \cdot y \in S$, which means that S is an ideal in $\mathfrak{A}$. □

4. Conclusion

In a way, this paper is a continuation of research on WS-algebras started by reports [4], [2] and [6]. In it, the author, besides describing some new properties of these algebras that were not known before, discusses about ideals in WS-algebras and especially p-ideals and implicative ideals in them. Additionally, apart from the previous one, the paper analyzes the atoms of this class of logical algebras. What could be done in the future, as a continuation of this research, is to register the existence and describe the properties of some other types of ideals in WS-algebras.

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