CREAT. MATH. INFORM.
Volume **34** (2025), No. 1,
Pages 81-88

Online version at https://creative-mathematics.cunbm.utcluj.ro/
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2025.01.08

On Duo Γ-semihyperrings

JITENDRASING J. PATIL[1] AND KISHOR F. PAWAR[2]

ABSTRACT. A characterization of a regular Γ-semihyperring to be a duo Γ-semihyperring with the help of ideals in Γ-semihyperrings has been taken into account. Also several ideal-theoretic characterizations of a regular duo Γ-semihyperring has been done and proved some important results in this respect.

1. INTRODUCTION AND PRELIMINARIES

In 1958, Feller [2] introduced the notion of duo ring. In [10] Thierrin studied some important properties of duo ring and he proved the set of nilpotent elements of a duo ring R is an ideal, which is the intersection of the completely prime ideals of R. A several ideal-theoretic characterizations of regular duo rings and semigroups are given by Lajos[3]. In [4], Lajos studied regular duo semigroups and proved some important results in this respect.

The notion of hypergroup was introduced by Marty in 1934. After that, many authors studied algebraic hyperstructure which are generalization of classical algebraic structure. In classical algebraic structure the composition of two elements is an element while in an algebraic hyperstructure composition of two elements is a set. The hyperstructure theory has lot of scope for study in the area of computer science and information technology nowadays. The theory of hyperstructure has vast applications in various streams of sciences. In 2003, Corsini and Leoreanu [1] has given application of theory of hyperstructure in geometry, cryptography, artificial intelligence, relation algebra, automata, median algebras, relation algebras, fuzzy sets and codes. So due to the vast applicability of the filed of hyperstructure it becomes important to study the different notions of classical algebraic structure to hyperstructure theory. Our main objective of the given paper is to study the notions duo rings and regular rings from classical algebraic structure to hyperstructure theory briefly.

The notion of Γ-semihyperrings as generalization of a semiring, a semihyperring and a Γ-semiring was introduced by Dehkordi and Davvaz [11]. Also, Pawar et al.[9] introduced regular Γ-semihyperrings and made it's characterization with the help of ideals in Γ-semihyperrings. In [7, 8] Patil and Pawar studied prime, semiprime ideals in Γ-semihyperrings and uniformly strongly prime Γ-semihyperrings briefly. As there is lot of scope to study different structure of classical algebraic structure in Γ-semihyperrings. So here it our attempt to study duo ring and regular structure of classical algebraic structure in Γ-semihyperrings. In this paper characterization of regular Γ-semihyperring R to be a

Received: 02.09.2023. In revised form: 15.09.2024. Accepted: 22.09.2024
2010 *Mathematics Subject Classification.* 16Y99, 16D25, 20N20.
Key words and phrases. *Quasi-Ideals, Bi-Ideals, Γ-semihyperrings and regular Γ-semihyperrings.*
Corresponding author: Jitendrasing J. Patil; jjpatil.777@rediffmail.com

duo Γ-semihperring has been done with the help of ideals in R. Later we made characterization of regular duo Γ-semihyperring and proved some results on the line of [4].

Here are some useful definitions and the readers are requested to refer [11].

Definition 1.1. *Let R be a commutative semihypergroup and Γ be a commutative group. Then, R is called a Γ-semihyperring if there is a map $R \times \Gamma \times R \to \wp^*(R)$ (images to be denoted by $a\alpha b$, for all $a, b \in R$ and $\alpha \in \Gamma$) and $\wp^*(R)$ is the set of all non-empty subsets of R satisfying the following conditions:*

(1) $a\alpha(b + c) = a\alpha b + a\alpha c$
(2) $(a + b)\alpha c = a\alpha c + b\alpha c$
(3) $a(\alpha + \beta)c = a\alpha c + a\beta c$
(4) $a\alpha(b\beta c) = (a\alpha b)\beta c$, *for all* $a, b, c \in R$ *and for all* $\alpha, \beta \in \Gamma$.

Definition 1.2. *A Γ-semihyperring R is said to be commutative if $a\alpha b = b\alpha a$, for all $a, b \in R$ and $\alpha \in \Gamma$.*

Definition 1.3. *A Γ-semihyperring R is said to be with zero, if there exists $0 \in R$ such that $a \in a + 0$ and $0 \in 0\alpha a, 0 \in a\alpha 0$, for all $a \in R$ and $\alpha \in \Gamma$.*

Let A and B be two non-empty subsets of a Γ-semihyperring R and $x \in R$, then

$$A + B = \{x | x \in a + b, a \in A, b \in B\}$$
$$A\Gamma B = \{x | x \in a\alpha b, a \in A, b \in B, \alpha \in \Gamma\}.$$

Definition 1.4. *A non-empty subset R_1 of Γ-semihyperring R is called a Γ-sub semihyperring if it is closed with respect to the multiplication and addition, that is, $R_1 + R_1 \subseteq R_1$ and $R_1\Gamma R_1 \subseteq R_1$.*

Definition 1.5. *A right (left) ideal I of a Γ-semihyperring R is an additive sub semihypergroup of $(R, +)$ such that $I\Gamma R \subseteq I(R\Gamma I \subseteq I)$. If I is both right and left ideal of R, then we say that I is a two sided ideal or simply an ideal of R.*

2. Duo Γ-semihyperrings

In this section, we have made characterization of a regular Γ-semihyperring to be a duo Γ-semihyperring. We have proved some results in this respect.

Definition 2.6. *A Γ-semihyperring R is said to be a left (right) duo Γ-semihyperring if every left (right) ideal of R is a right (left) ideal of R.*

Definition 2.7. *A Γ-semihyperring R is said to be a duo Γ-semihyperring if any one sided ideal of R is a two sided ideal of R. That is a Γ-semihyperring R is said to be a duo Γ-semihypering if it is a left as well as right duo Γ-semihyperring.*

Example 2.1. [11] *Let $R = \{a, b, c, d\}, \Gamma = \mathbb{Z}_2$ and $\alpha = \bar{0}, \beta = \bar{1}$. Then R is a Γ-semihyperring with the following hyperoperations*

$+$	a	b	c	d
a	$\{a,b\}$	$\{a,b\}$	$\{c,d\}$	$\{c,d\}$
b	$\{a,b\}$	$\{a,b\}$	$\{c,d\}$	$\{c,d\}$
c	$\{c,d\}$	$\{c,d\}$	$\{a,b\}$	$\{a,b\}$
d	$\{c,d\}$	$\{c,d\}$	$\{c,d\}$	$\{a,b\}$

β	a	b	c	d
a	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$
b	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$
c	$\{a,b\}$	$\{a,b\}$	$\{c,d\}$	$\{c,d\}$
d	$\{a,b\}$	$\{a,b\}$	$\{c,d\}$	$\{c,d\}$

For any $x, y \in R$ we define $x\alpha y = \{a, b\}$. Here R is a duo Γ-semihyperring.

Definition 2.8. [5] *A subsemihypergroup Q of $(R, +)$ is said to be a quasi-ideal of Γ-semihyperring R if $(R\Gamma Q) \cap (Q\Gamma R) \subseteq Q$.*

Definition 2.9. [5] *An element e of Γ-semihyperring R is said to be a left (right) identity of R if $r \in e\alpha r (r \in r\alpha e)$, for all $r \in R$ and $\alpha \in \Gamma$. An element e of Γ-semihyperring R is said to be a two sided identity or simply an identity if e is both left and right identity, that is $r \in e\alpha r \cap r\alpha e$, for all $r \in R$ and $\alpha \in \Gamma$.*

Definition 2.10. [6] *A non-empty set B of a Γ-semihyperring R is a bi-ideal of R if B is a Γ-subsemihyperring of R and $B\Gamma R\Gamma B \subseteq B$.*

Definition 2.11. [9] *A subset A of a Γ-semihyperring R is said to be a regular (strongly regular) if there exists $\Gamma_1, \Gamma_2 \subseteq \Gamma$ and $B \subseteq R$ such that $A \subseteq A\Gamma_1 B\Gamma_2 A$ ($A = A\Gamma_1 B\Gamma_2 A$).*

A singleton set $\{a\}$ of a Γ-semihyperring is regular if there exists $\Gamma_1, \Gamma_2 \subseteq \Gamma$ and $B \subseteq R$ such that

$$\{a\} = a \in a\Gamma_1 B\Gamma_2 a = \{x \in R | x \in a\alpha b\beta a, \alpha \in \Gamma_1, \beta \in \Gamma_2, b \in B\}.$$

That is a singleton set $\{a\}$ of Γ-semihyperring is a regular if there exists $\alpha, \beta \in \Gamma, b \in R$ such that $a \in a\alpha b\beta a$. Similarly, a singleton set $\{a\}$ of Γ-semihyperring is a strongly regular if there exists $\alpha, \beta \in \Gamma, b \in R$ such that $\{a\} = a = a\alpha b\beta a$. Simply we can say a element $a \in R$ is regular (strongly regular) instead of a singleton set $\{a\}$ of a Γ-semihyperring is a regular(strongly regular).

Definition 2.12. [9] *A Γ-semihyperring R is said to be a regular (strongly regular) if every element of R is a regular (strongly regular).*

Example 2.2. [9] *Consider the following:*

$$R = \{\begin{pmatrix} x & y \\ z & w \end{pmatrix} | x, y, z, w \in \mathbb{R}\},$$
$$\Gamma = \{z | z \in \mathbb{Z}\},$$
$$A_\alpha = \{\begin{pmatrix} \alpha a & 0 \\ 0 & \alpha b \end{pmatrix} | a, b \in \mathbb{R}, \alpha \in \Gamma\}.$$

Then R is a Γ-semihyperring under the matrix addition with hyperoperation $M\alpha N \to MA_\alpha N$, for all $M, N \in R$ and $\alpha \in \Gamma$. It is easy to verify, R is not a duo as well as not regular Γ-semihyperring.

Theorem 2.1. [9] *If R be a Γ-semihyperring with an identity. Then R is a regular if and only if for any left ideal A and right ideal B of R, $A \cap B = B\Gamma A$.*

Theorem 2.2. [5] *Any one sided ideal or two sided ideal of a Γ-semihyperring R is a quasi-ideal of Γ-semihyperring R.*

Theorem 2.3. [5] *If Γ-semihyperring R has an identity element e then every quasi-ideal of a Γ-semihyperring R can be expressed as an intersection of a left ideal and a right ideal of a Γ-semihyperring R.*

Theorem 2.4. [6] *Any one sided (two sided) ideal of a Γ-semihyperring R is a bi-ideal of R.*

Theorem 2.5. [6] *An ideal B of a Γ-semihyperring R is a bi-ideal of R if and only if there exists a left ideal P and right ideal Q of R such that $Q\Gamma P \subseteq B \subseteq P \cap Q$.*

Theorem 2.6. [6] *Every quasi-ideal of a Γ-semihyperring R is a bi-ideal of R.*

Theorem 2.7. *Let R be a regular Γ-semihyperring with an identity. Then R is a left duo Γ-semihyperring if and only if for any two left ideals A and B of R, $A \cap B = A\Gamma B$.*

Proof. Let R be a left duo regular Γ-semihyperring with an identity and A, B be any two left ideals of R. Then A is a right ideal of R. Then by Theorem 2.1, we have $A \cap B = A\Gamma B$. Conversely, assume that for any two left ideals A and B of R, $A \cap B = A\Gamma B$ and L be a any left ideal of R. Hence by assumption, $L\Gamma R = L \cap R = L$. So we get, L is a right ideal of R. Therefore R is a left duo Γ-semiring. □

Theorem 2.8. *Let R be a regular Γ-semihyperring with an identity. Then R is a right duo Γ-semihyperring if and only if for any two right ideals A and B of R, $A \cap B = A\Gamma B$.*

Example 2.3. [9] *If $R = \{a, b, c, d\}$ then R is a commutative semihypergroup with following hyperoperations:*

$+$	a	b	c	d
a	$\{a\}$	$\{a,b\}$	$\{a,c\}$	$\{a,d\}$
b	$\{a,b\}$	$\{b\}$	$\{b,c\}$	$\{b,d\}$
c	$\{a,c\}$	$\{b,c\}$	$\{c\}$	$\{c,d\}$
d	$\{a,d\}$	$\{b,d\}$	$\{c,d\}$	$\{d\}$

$\cdot$	a	b	c	d
a	$\{a\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,c,d\}$
b	$\{a,b\}$	$\{b\}$	$\{b,c\}$	$\{b,c,d\}$
c	$\{a,b,c\}$	$\{b,c\}$	$\{c\}$	$\{c,d\}$
d	$\{a,b,c,d\}$	$\{b,c,d\}$	$\{c,d\}$	$\{d\}$

Then R is a regular as well as duo Γ-semihyperring with the operation $x\alpha y \to x \cdot y$, for $x, y \in R$ and $\alpha \in \Gamma$, where Γ-is any commutative group.

Theorem 2.9. *Let R be a Γ-semihyperring with an identity element. Then R is a left duo Γ-semihyperring if and only if every quasi-ideal of R is a right ideal of R.*

Proof. Let R be a left duo Γ-semihyperring with an identity and Q be any quasi-ideal of R. Then by Theorem 2.3, there exists a left ideal A and a right ideal B such that $Q = A \cap B$. Since R is a left duo Γ-semihyperring, left ideal A is a right ideal of R. Hence we get, $Q = A \cap B$ is a right ideal of R.
Conversely, assume that every quasi-ideal of R is a right ideal of R. By Theorem 2.2, any left ideal L of R is a quasi-ideal of R. Hence by Assumption, L is a right ideal of R. This shows that, R is a left duo Γ-semihyperring. □

Theorem 2.10. *Let R be a Γ-semihyperring with an identity element. Then R is a right duo Γ-semihyperring if and only if every quasi-ideal of R is a left ideal of R.*

Corollary 2.1. *Let R be a Γ-semihyperring with an identity element. Then R is a duo Γ-semihyperring if and only if every quasi-ideal of R is an ideal of R.*

Example 2.4. *In Example 2.1, R is a duo Γ-semihyperring. Then by above Theorem quasi-ideal $Q = \{a, b\}$ of R being an ideal of R.*

Theorem 2.11. *Let R be a regular Γ-semihyperring with an identity. Then R is a right duo Γ-semihyperring if and only if every bi-ideal of R is a left ideal of R.*

Proof. Let R be a right duo Γ-semihyperring with an identity and B be any bi-ideal of R. Then by Theorem 2.5, there exists a left ideal P and a right ideal Q of R such that $Q\Gamma P \subseteq B \subseteq P \cap Q$. Since R is a Γ-semihyperring with identity by Theorem 2.1, $Q\Gamma P = P \cap Q = B$. As R is a right duo Γ-semihyperring so right ideal Q is a left ideal of R. Therefore we get, $P \cap Q = B$ is a left ideal of R.
Conversely, assume that every bi-ideal of R is a left ideal of R. If B be a right ideal of R.

Then by Theorem 2.4, B is a bi-ideal of R. Hence by assumption, B is a left ideal of R. Therefore we get, R is a right duo Γ-semihyperring. □

Theorem 2.12. *Let R be a regular Γ-semihyperring with an identity. Then R is a left duo if and only if every bi-ideal of R is a right ideal of R.*

Corollary 2.2. *Let R be a regular Γ-semihyperring with an identity. Then R is duo Γ-semihyperring if and only if every bi-ideal of R is an ideal of R.*

3. REGULAR DUO Γ-SEMIHYPERRINGS

In this section, we have made several ideal-theoretic characterization of a regular duo Γ-semihyperring on the line of [4].

Theorem 3.13. *Let B be a bi-ideal of Γ-semihyperring R. Then $(B \cup B\Gamma R)$ is a smallest right ideal of R containing B and $(B \cup R\Gamma B)$ is a smallest left ideal of R containing B.*

Proof. Let B be a bi-ideal of Γ-semihyperring R. Then B is a Γ-subsemihyperring of R. So we get, $(B \cup B\Gamma R) + (B \cup B\Gamma R) = (B + B) \cup (B + B\Gamma R) \cup (B\Gamma R + B\Gamma R) \cup (B\Gamma R + B) \subseteq (B \cup B\Gamma R)$ i.e. $(B \cup B\Gamma R)$ is an additive sub-semihypergroup of $(R, +)$ and $(B \cup B\Gamma R)\Gamma R = (B\Gamma R) \cup (B\Gamma R\Gamma R) \subseteq (B \cup B\Gamma R)$.Therefore we get, $(B \cup B\Gamma R)$ is a right ideal of R.
Now, claim to show $(B \cup B\Gamma R)$ is a smallest right ideal of R containing B. Assume that, $B^{'}$ is a smallest right ideal of R containing B. Then $B^{'} \subseteq (B \cup B\Gamma R)$. Also, $B \subseteq B^{'}$ and $B\Gamma R \subseteq B^{'}\Gamma R \subseteq B^{'}$ since $B^{'}$ is a right ideal of R. Thus we get $B^{'} \subseteq (B \cup B\Gamma R) \subseteq B^{'}$. Therefore we get, $B^{'} = (B \cup B\Gamma R)$ i.e. $(B \cup B\Gamma R)$ is a smallest right ideal of R containing B.
Similarly we can show that, $(B \cup R\Gamma B)$ is a smallest left ideal of R containing B. □

Corollary 3.3. *Let I be any one sided (two sided) ideal of a Γ-semihyperring R. Then $(I \cup I\Gamma R)$ is a smallest right ideal of R containing I and $(I \cup R\Gamma I)$ is a smallest left ideal of R containing I.*

Proof. Proof easily follows by Theorem 2.4. □

Theorem 3.14. *A non-empty subset B of a regular Γ-semihyperring R is a bi-ideal of R if and only if there exists a left ideal L and a right ideal M of R such that $B = M\Gamma L$.*

Proof. Let B be a bi-ideal of a regular Γ-semihyperring R. Consider K is the product of the smallest right ideal of a R containing B and the smallest left ideal of a R containing B i.e $K = (B \cup B\Gamma R)\Gamma(B \cup R\Gamma B)$. Claim to show $B = K$. Clearly, $K = (B \cup B\Gamma R)\Gamma(B \cup R\Gamma B) = (B\Gamma B) \cup (B\Gamma R\Gamma B)$. Since R is a regular Γ-semihyperring so for any $b \in B$, $b \in b\Gamma R\Gamma b \subseteq B\Gamma R\Gamma B \subseteq K$. Thus we get, $B \subseteq K$. Also, we have $B\Gamma B \subseteq B$ and $B\Gamma R\Gamma B \subseteq B$ since B is a bi-ideal of R. Therefore we get, $K = (B\Gamma B) \cup (B\Gamma R\Gamma B) \subseteq B$. Thus $B \subseteq K$ and $K \subseteq B$ implies that $B = (B\Gamma B) \cup (B\Gamma R\Gamma B) = K$.
Conversely, assume that L be a left ideal of and M be a right ideal of a Γ-semihyperring R such that $B = M\Gamma L$. Then $(M\Gamma L) + (M\Gamma L) \subseteq (M\Gamma L)$ and $(M\Gamma L)\Gamma(M\Gamma L) \subseteq (M\Gamma L)$. Therefore we get, $(M\Gamma L)$ is a Γ-subsemihyperring of R. Also $(M\Gamma L)\Gamma R\Gamma(M\Gamma L) \subseteq M\Gamma L$ i.e $M\Gamma L = B$ is a bi-ideal of Γ-semihyperring R. □

Theorem 3.15. *Every bi-ideal of a regular duo Γ-semihyperring is a two sided ideal.*

Proof. Let R be a regular duo Γ-semihyperring. Then by Theorem 3.14, there exists a left ideal L and a right ideal M of R such that $B = M\Gamma L$. Since product of two sided is a two sided ideal result follows. □

Corollary 3.4. *Every quasi-ideal of a regular duo Γ-semihyperring is a two sided ideal.*

Proof. Since every quasi-ideal is a bi-ideal of Γ-semihyperring result easily follows. □

Theorem 3.16. *Let R be a Γ-semihyperring with an identity. Then R is a regular duo Γ-semihyperring if and only if for any left ideals L and right ideal M of R $(L \cap L\Gamma R)^2 = L$ and $(M \cap R\Gamma M)^2 = M$.*

Proof. Let R be a regular duo Γ-semihyperring with an identity. Then for any left ideal of R we can write, $(L \cap L\Gamma R) = L$. By Theorem 2.1, for ideal $(L \cap L\Gamma R)$ we get, $(L \cap L\Gamma R)\Gamma(L \cap L\Gamma R) = (L \cap L\Gamma R)^2 = (L \cap L\Gamma R) \cap (L \cap L\Gamma R) = (L \cap L\Gamma R) = L$. Similarly we can show that, $(M \cap R\Gamma M)^2 = M$.
Conversely, assume that $(L \cap L\Gamma R)^2 = L$ and $(M \cap R\Gamma M)^2 = M$ for any left ideal L and right ideal M of R. So clearly it gives every right ideal of R is left ideal of R and vice versa i.e.we get, R is a duo Γ-semihyperring. Then by assumption for any ideal I of R we have, $I\Gamma I = I$. Now, By Theorem 2.1, R is a regular Γ-semihyperring. Hence the converse. □

Theorem 3.17. *Let R be a Γ-semihyperring with an identity. Then R is a regular duo Γ-semihyperring if and only if for any bi-ideal B of R, $(B \cap B\Gamma R)^2 = B = (B \cap R\Gamma B)^2$.*

Proof. By theorems 3.15 and 3.4 proof easily follows. □

Corollary 3.5. *Let R be a Γ-semihyperring with an identity. Then R is a regular duo Γ-semihyperring if and only if for any quasi-ideal Q of R, $(Q \cap Q\Gamma R)^2 = Q = (Q \cap R\Gamma Q)^2$.*

Theorem 3.18. *In a Γ-semihyperring R with an identity following conditions are equivalent.*

(1) R is regular duo Γ-semihyperring.
(2) For any two quasi-ideals Q_1 and Q_2 of R, $Q_1 \cap Q_2 = Q_1\Gamma Q_2$.
(3) For a left ideal A and a right ideal B of R, $A \cap B = A\Gamma B$.
(4) For each left ideals A_1, A_2 and right ideals B_1, B_2 of R, $A_1 \cap A_2 = A_1\Gamma A_2$ and $B_1\Gamma B_2$.

Proof. $(1) \Rightarrow (2)$
Let Q_1 and Q_2 be any two quasi-ideals of a regular duo Γ-semihyperring R. Then by Corollary 3.4, Q_1 and Q_2 are the ideals of R. Since R is regular by Theorem 2.1, we get $Q_1 \cap Q_2 = Q_1\Gamma Q_2$.
$(2) \Rightarrow (3)$
By Theorem 2.2, implication follows easily.
$(3) \Rightarrow (1)$
Suppose that for a left ideal A and a right ideal B of R, $A \cap B = A\Gamma B$. Then by Theorem 2.1, R is a regular. Now for $B = R$, we have $A\Gamma R = A \cap R = A$. This shows that left ideal A of R is a right ideal of R. Similarly for $A = R$ we get right ideal B is a left ideal of R. Therefore R is duo Γ-semihyperring i.e. R is a regular duo Γ-semihyperring.
$(2) \Rightarrow (4)$
By Theorem 2.2, implication follows easily.
$(4) \Rightarrow (1)$
Let A be any left ideal of R. Then by assumption, $A\Gamma R = A \cap R = A$ as R is itself left ideal. Therefore any left ideal A of R is right ideal of R. Similarly, we can show any right ideal is a left ideal of R. Therefore R is duo Γ-semihyperring. We can easily show R is regular. □

Theorem 3.19. *In a Γ-semihyperring R with an identity following conditions are equivalent.*

(1) R is a regular duo Γ-semihyperring.
(2) For every ideal I and bi-ideal B of R, $I \cap B = I\Gamma B\Gamma I$.
(3) For every ideal I and quasi-ideal Q of R, $I \cap Q = I\Gamma Q\Gamma I$.

Proof. $(1) \Rightarrow (2)$
Let R be a regular duo Γ-semihyperring, I be an ideal and B be a bi-ideal of R. Then by Theorem 3.15, we have B is an ideal of R. Therefore $I\Gamma B\Gamma I \subseteq I$ and $I\Gamma B\Gamma I \subseteq B$, since

I and B are ideal of R. Therefore we get, $I\Gamma B\Gamma I \subseteq I \cap B$. For the reverse inclusion, let $a \in I \cap B$. Then $a \in a\Gamma R\Gamma a$, since R is regular.

$$\begin{aligned} a\Gamma R\Gamma a &\subseteq a\Gamma R\Gamma(a\Gamma R\Gamma a) && \text{(since } a \in a\Gamma R\Gamma a\text{)} \\ &= (a\Gamma R)\Gamma a\Gamma(R\Gamma a) \\ &\subseteq (I\Gamma R)\Gamma B\Gamma(R\Gamma I) && \text{(since } a \in I \cap B\text{)} \\ &\subseteq I\Gamma B\Gamma I. && \text{(since } I \text{ is an ideal)} \end{aligned}$$

Therefore $a \in a\Gamma R\Gamma a \subseteq I\Gamma B\Gamma I$. Thus $a \in I \cap B$ implies that $a \in I\Gamma B\Gamma I$. This shows that $I \cap B \subseteq I\Gamma B\Gamma I$. Hence we get $I \cap B = I\Gamma B\Gamma I$.
$(2) \Rightarrow (3)$
As every quasi-ideal of R is a bi-ideal of R implication follows.
$(3) \Rightarrow (1)$
Let A be a left ideal of Γ-semihyperring R. The by Theorem 2.2, A is a quasi-ideal of R. Since R is an ideal and A is a quasi-ideal of R, by assumption we have $A = R \cap A = R\Gamma A\Gamma R$. $A\Gamma R = (R\Gamma A\Gamma R)\Gamma R = R\Gamma A\Gamma R\Gamma R \subseteq R\Gamma A\Gamma R = A$. Thus we get $A\Gamma R \subseteq A$ gives A is a right ideal of R. Similarly we can show any right ideal of R is a left ideal of R. Therefore R is a duo Γ-semihyperring. Now as any left ideal A and right ideal B are two sided ideal of R, by assumption $B \cap A = B\Gamma A\Gamma B$. But $B \cap A = B\Gamma A\Gamma B \subseteq B\Gamma A$, since A is a ideal. Thus we get $B \cap A \subseteq B\Gamma A$. Also, $B\Gamma A \subseteq B \cap A$, since A and B are ideals. We get, $B \cap A = B\Gamma A$. Hence By Theorem 2.1, R is a regular Γ-semihyperring. Thus we get, R is a regular duo. □

Theorem 3.20. *Let R be a Γ-semihyperring with an identity element. Then R is a regular duo Γ-semihyperring if and only if $A \cap B = A\Gamma B\Gamma R$, for a left ideal A and a right ideal B of R.*

Proof. Let R be a regular duo Γ-semihyperring with identity element and A be a left ideal and B be a right ideal of R. Then $A\Gamma B\Gamma R \subseteq A \cap B$ since A and B are the ideals of R. For the reverse inclusion, let $a \in A \cap B$. Then as R is regular, $a \in a\Gamma R\Gamma a$. Now

$$\begin{aligned} a\Gamma R\Gamma a &\subseteq (a\Gamma R\Gamma a)\Gamma R\Gamma a && \text{(since } a \in a\Gamma R\Gamma a\text{)} \\ &\subseteq (A\Gamma R\Gamma B)\Gamma R && \text{(since } a \in A \cap B\text{)} \\ &= (A\Gamma R)\Gamma B\Gamma R \subseteq A\Gamma B\Gamma R. \end{aligned}$$

Thus $A \cap B \subseteq A\Gamma B\Gamma R$. Hence we get, $A \cap B = A\Gamma B\Gamma R$.
Conversely, let A be a left ideal of R. Then by assumption, we have $A = A \cap R = A\Gamma R\Gamma R$. This shows that A is right ideal of R. Similarly we can show any right ideal of R is left ideal of R. Therefore R is a duo Γ-semihyperring. By using assumption and R is a duo we can easily show, $A \cap B = A\Gamma B$. By Theorem 2.1, we get R is regular. Therefore we get, R is a regular duo. □

Theorem 3.21. *Let R be a Γ-semihyperring with an identity element. Then R is a regular duo Γ-semihyperring if and only if $A \cap B = R\Gamma A\Gamma B$, for a left ideal A and a right ideal B of R.*

Theorem 3.22. *Let R be a Γ-semihyperring with identity element. Then R is a regular duo if and only if $A \cap I = A\Gamma I$ and $B \cap I = I\Gamma B$, for a left ideal A, a right ideal B and an ideal I of R.*

Proof. Let R be a regular duo Γ-semihyperring with identity element and A be a left ideal and I be a ideal of R. Then we easily get, $A\Gamma I \subseteq A \cap I$. For the reverse inclusion, let $a \in A\Gamma I$. As R is a regular Γ-semihyperring, we have $a \in a\Gamma R\Gamma a$. Then

$$\begin{aligned} a \in a\Gamma R\Gamma a &\subseteq A\Gamma R\Gamma I && \text{(since } a \in A \cap I\text{)} \\ &= A\Gamma(R\Gamma I) \\ &= A\Gamma I. && \text{(since } I \text{ is an ideal of } R\text{)} \end{aligned}$$

Thus $a \in A \cap I$ implies $a \in A\Gamma I$. Therefore we get, $A \cap I = A\Gamma I$. Similarly, we can show $B \cap I = I\Gamma B$.
Conversely, let A be a left ideal of R. By assumption $A = A \cap R = A\Gamma R$, which shows A is right ideal of R. Similarly, we can show any right ideal of R is left left ideal. Therefore R is a duo Γ-semihyperring. Now, let A is a left ideal and B is a right ideal of R. As R is duo Γ-semihyperring by assumption we have $A \cap B = B\Gamma A$. Therefore By Theorem 2.1, R is a regular Γ-semihyperring. □

4. Conclusion

The hyperstructure theory has the vast application various fields of sciences. So it is essential to study the concepts of classical algebraic structure to algebraic hyperstructure. In this paper we studied the concepts of duo Γ-semihyperrings and regular duo Γ-semihyperrings. Also we made characterizations of duo Γ-semihyperrings and regular duo Γ-semihyperrings with the help of different types of ideals in the Γ-semihyperrings. Also, there is scope to made characterizations of duo Γ-semihyperrings and regular duo Γ-semihyperrings with the help of fuzzy ideals in Γ-semihyperrings. As fuzzy sets plays crucial role in application of computer sciences and artificial intelligence there is lot of scope to make advanced study in the same concept.

5. Acknowledgments

The Authors express their gratitude to anonymous referees for useful comments and suggestions that improved the present paper.

References

[1] Corsini, P.; Leoreannu, L. *Application of Hyperstructure Theory*. Kluwr, Dordrecht, 2003.
[2] Feller, E. Properties of Primary Non Commutative Rings. *Trans. Amer. Math. Soc.* **89** (1958), 79-91.
[3] Lajos, S. Some Characterizations of Regular Duo Rings and Semigroups. *Proc. Japan Acad. Ser. A Math. Sci.* **46** (1970), no. 10, 1099-1101.
[4] Lajos, S. A new Characterizations of Regular Duo Semigroups. *Proc. Japan Acad. Ser. A Math. Sci.* **47** (1971), no. 4, 394-395.
[5] Patil, J.; Pawar, K. On quasi-ideals of Γ-semihyperrings. *J. Hyperstruct.* **8** (2019), no.2, 123-134.
[6] Patil, J.; Pawar, K. On bi-ideals of Γ-semihyperrings. *J. Hyperstruct.* **11** (2022), no.2, 183-195.
[7] Patil, J.; Pawar, K. On Prime and Semiprime Ideals of Γ- semihyperrings. *J. Algebra Relat. Topics* **9** (2021), no. 1, 131-141.
[8] Patil, J.; Pawar, K. On Uniformly Strongly Prime Γ- semihyperring. *Kragujevac J. Math.* **48** (2024), no. 1, 79-87.
[9] Pawar, K.; Patil, J.; Davvaz, B. On regular Γ- semihyperrings and idempotent Γ- semihyperrings. *Kyungpook Math. J.* **59** (2019), 35-45.
[10] Thierrin, G. On Duo Rings. *Canad. Math. Bull.* **3** (1960), no.2, 167-172.
[11] Ostadhadi-Dehkordi, S.; Davvaz, B. Ideal theory in Γ- semihyperrings. *Iran. J. Sci. Technol. Trans. A Sci.* **37** (2013), no.3, 251-263.

[1] Department of Mathematics, Indraraj Arts, Commerce and Science college, Sillod, Affiliated to Dr. Babashasheb Ambedkar Marathwada University Aurangabad- 431 112, India.
Email address: `jjpatil.777@rediffmail.com`

[2] Department of Mathematics, School of Mathematical Sciences, Kavayitri Bahinabai Chaudhari North Maharashtra University, Jalgaon - 425 001, India.
Email address: `E-kfpawar@nmu.ac.in`