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Isomorphism of Line Graph and Common Neighbourhood Graph

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ABSTRACT. The line graph $L(G)$ of a graph G has the edges of G as its vertices and two distinct edges of G are adjacent in $L(G)$ if they are incident in G. The common neighborhood graph (congraph) of G, denoted by $con(G)$, is the graph with vertex set $\{v_1, v_2, \dots, v_n\}$, in which two vertices are adjacent if and only if they have at least one common neighbor in the graph G. In this paper we characterize the graphs G for which $L(G)$ and $con(G)$ are isomorphic.

1. INTRODUCTION

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. If there is no ambiguity in the choice of G, then we write $V(G)$ and $E(G)$ as V and E respectively. A discrete dynamic system is any set Γ together with a mapping $\phi : \Gamma \to \Gamma$, called a graph operator. Operations on graphs are a method by which we seek to construct new graphs from a set of graphs. There are so many operators out of which the first main operator is the line graph operator. The line graph $L(G)$ of a graph G has the edges of G as its vertices and two distinct edges of G are adjacent in $L(G)$ if they are incident in G [9]. Line graphs make important connections between many important areas of graph theory. For example, determining a maximum matching in a graph is equivalent to finding a maximum independent set in the corresponding line graph. Similarly, edge colouring is equivalent to vertex colouring in the line graph. Much research has been done on the study and application of line graphs; a comprehensive survey of results is found in[10].

Let G be a simple graph with vertex set $\{v_1, v_2, \dots, v_n\}$. The common neighborhood graph(congraph) of G, denoted by $con(G)$, is the graph with vertex set $\{v_1, v_2, \dots, v_n\}$, in which two vertices are adjacent if and only if they have at least one common neighbor in the graph G [1]. Some interesting papers on the line graph and common neighbourhood graph are [2], [3], [5] and [6].

1.1. Basic Definitions and Preliminaries.

A complete graph on n vertices, denoted by K_n, is the graph in which any two vertices are adjacent. A path on n vertices, denoted by P_n, is the graph with vertex set $\{v_1, v_2, \dots, v_n\}$ and v_i is adjacent to v_{i+1} for $i = 1, 2, \dots, n-1$. If in addition, v_n is adjacent to v_1 and $n \geq 3$, it is called a cycle of length n, denoted by C_n. A graph is said to be unicyclic if it has exactly one cycle. Caterpillar is a tree in which all the vertices are within distance one of a central path. ie, if we remove all the pendant vertices from a caterpillar, the resulting graph will be a path. $G \circ H$ be the coalescence in which one vertex u of G is identified with one vertex v of H.

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Two operators A and B are said to be isomorphic if and only if $A(G) \cong B(G)$ for every graph G. Generally two operators are not isomrphic, but we can characterize the graphs for which they are isomorphic. Such study is included in papers [7] and [8].

All graph theoretic notations and terminology not mentioned here are from [2].

The following result is useful in our paper.

Theorem 1.1. [1] $con(C_n) \cong \begin{cases} C_n, \textit{if } n \textit{ is odd and } n \geq 3, \\ P_2 \cup P_2, \textit{ if } n = 4, \\ C_{\frac{n}{2}} \cup C_{\frac{n}{2}}, \textit{if } n \textit{ is even and } n \geq 6. \end{cases}$

2. Isomorphism of $L(G)$ and $con(G)$

Theorem 2.2. *For a connected graph G if $con(G) \cong L(G)$, then G is a unicyclic graph without even cycles.*

Proof. If G is a connected graph such that $con(G) \cong L(G)$, then $|V(G)| = |E(G)|$. Hence G is a unicyclic graph. If G contains an even cycle C_{2k}, then C_{2k} will make a C_{2k} itself in $L(G)$ while $P_k \cup P_k$, if $k = 2$ and $C_k \cup C_k$ if $k \geq 3$ in $con(G)$ which is a contradiction. Hence we conclude that G is a unicyclic graph without even cycles. □

Note: From Theorem 2.2 it is clear that the graphs with $con(G) \cong L(G)$ can be constructed from a tree by adding one edge to it. For convenience, in this paper we use the terminology 'root tree' for the initial tree.

Theorem 2.3. *If $con(G) \cong L(G)$, then the root tree is a caterpillar graph.*

Proof. Let G be a graph with $con(G) \cong L(G)$. If the root tree is not a caterpillar graph, there exists a non pendant vertex u which is adjacent to at least three non pendant vertices, say u_1, u_2 and u_3. Clearly, u, u_1, u_2 and u_3 form an induced $K_{1,3}$ with u as the central vertex. Since u_1, u_2 and u_3 are non pendant vertices there exist vertices v_1, v_2, v_3 adjacent to u_1, u_2, u_3 respectively. When we consider the subgraph induced by the vertices u, v_1, v_2 and v_3 in $con(G)$ there are two possibilities: the vertices u, v_1, v_2 and v_3 either form an induced $K_{1,3}$ or not. The first case is not possible since $K_{1,3}$ is a forbidden subgraph of $L(G)$. For the second case, there exists an edge connecting any two of the vertices in the set $\{v_1, v_2, v_3\}$. This is possible only when there exists a vertex p common to any two of the vertices in the set $\{v_1, v_2, v_3\}$. Without loss of generality assume p is a common vertex of v_1 and v_2. If $p \notin \{u, u_1, u_2, u_3, \}$, then the vertices $u, u_1, v_1, p, v_2, u_2, u$ form a cycle of length 6. By Theorem 2.2 this case is not possible in G. Also note that, since G is a unicyclic graph $p \neq u$ (Refer Figure 1). Now if $p \in \{u_1, u_2, u_3, \}$, then the vertices u, v_1, v_2 together with p induce a C_4 (Refer Figure 2). This also leads to a contradiction to Theorem 2.2. Therefore, if $con(G) \cong L(G)$, then in the root tree the non pendant vertices form a path. Hence G is a caterpillar graph. □

Theorem 2.4. *If G is a caterpillar graph where the non pendant vertices are $v_1, v_2, \ldots, v_k$ with degrees $d_1, d_2, \ldots d_k$, then,*

(1) $L(G) \cong K_{d_1} \circ K_{d_2} \circ \ldots \circ K_{d_k}$.

(2) $con(G) \cong G_1 \cup G_2$, *where* $G_1 \cong K_{d_1} \circ K_{d_3} \circ \ldots \circ K_{d_k}$, $G_2 \cong K_{d_2} \circ K_{d_4} \circ \cdots \circ K_{d_{k-1}}$ *when k is an odd number and* $G_1 \cong K_{d_1} \circ K_{d_3} \circ \cdots \circ K_{d_{k-1}}$, $G_2 \cong K_{d_2} \circ K_{d_4} \circ \cdots \circ K_{d_k}$ *when k is an even number .*

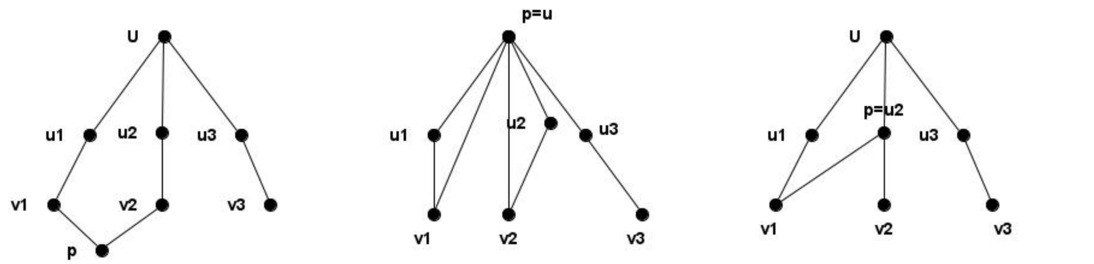

FIGURE 1

Proof. In G, since the number of edges incident on v_i is d_i, these d_i edges form a complete graph K_{d_i} in $L(G)$. The vertex corresponding to the edge $v_{i-1}v_i$ in G is the only vertex common to $K_{d_{i-1}}$ and K_{d_i} in $L(G)$. Therefore in $L(G)$, $K_{d_{i-1}}$ and K_{d_i} have exactly one common vertex. Hence $L(G) \cong K_{d_1} \circ K_{d_2} \circ \cdots \circ K_{d_k}$.
In G, the vertex v_i is a common neighbour of every vertices in $N(v_i)$. By Theorem 2.3, the only non pendant vertices adjacent to v_i are v_{i-1} and v_{i+1}. The vertex $v_{i-1} \in N(v_{i-2}) \cap N(v_i)$ and the vertex $v_{i+1} \in N(v_i) \cap N(v_{i+2})$. Therefore in $con(G)$, K_{d_i} and $k_{d_{i+2}}$ have exactly one common vertex. Hence $con(G)$ is the disjoint union of two components $K_{d_1} \circ K_{d_3} \circ \cdots \circ K_{d_k}$ and $K_{d_2} \circ K_{d_4} \circ \cdots \circ K_{d_{k-1}}$ when k is an odd number and $K_{d_1} \circ K_{d_3} \circ \cdots \circ K_{d_{k-1}}$ and $K_{d_2} \circ K_{d_4} \circ \cdots \circ K_{d_k}$ when k is an even number. □

Note
By Theorems 2.2 and 2.3, to find the graphs for which $L(G) \cong con(G)$ we have to concentrate only on the collection of graphs, $\mathcal{G} = \{T \cup e,\ where\ T\ is\ a\ caterpillar\ graph\}$.

Theorem 2.5. *In the class of graphs, $\mathcal{G} = \{T \cup e,\ where\ T\ is\ a\ caterpillar\ graph\}$,*

(1) if e is an edge joing two non pendant vertices v_l and v_m, $con(G) \cong L(G)$ if and only if l and m are odd numbers and the sequence $d_1 + 1, d_2, d_3, \ldots, d_{m-1}, d_m + 1$ is a cyclic permutation of $d_1 + 1, d_3, d_5, \ldots, d_{m-2}, d_m + 1, d_2, d_4, \ldots, d_{m-1}$;
(2) if $e = u_l v_m$, where u_l is a pendant vertex and v_m is a non pendant vertex, $con(G) \cong L(G)$ if and only if m is even and l odd and the sequence $d_1, d_m{+}1, d_{m-2}, d_{m-4}, \ldots, d_2, 2, d_{m-1}, \ldots, d_3$ is a cyclic permutation of $d_1, d_2, d_3, \ldots, d_{m-1}, d_m + 1, 2$;
(3) $e = u_l u_m$, where u_l and u_m are pendant vertices. Let the nonpendant vertices adjacent to u_l, u_m are v_l, v_m respectively then, $con(G) \cong L(G)$ if and only if
(1) T is a star graph, when $v_l = v_m$
(2) l and m are odd and the sequence $d_2, d_4, d_6, \ldots d_{m-1}, 2, d_1, d_3, \ldots, d_m, 2$ is a cyclic permutation of $d_1, d_2, d_3, \ldots, d_{m-1}, d_m, 2, 2$.

where $v_1, v_2, \ldots, v_k$ denote the non pendant vertices of T and $d_1, d_2, \ldots, d_k$ denote their degrees respectively.

Proof. Consider a caterpillar graph T with non pendant vertices $v_1, v_2, \ldots, v_k$ with degrees $d_1, d_2, \ldots, d_k$ respectively. While adding a new edge to a caterpillar graph, we have to consider the following cases.

(1) Both are non pendant vertices.
(2) Exactly one is a pendant vertex.
(3) Both are pendant vertices.

Case 1: Let $e = v_l v_m$. By Theorem 2.2, since even cycles are forbidden in G, l and m are either both even or both odd.

If l and m are even numbers (or odd numbers) then both the vertices v_l and v_m belong to G_1 (or G_2) in $con(T)$. If we join v_l and v_m by an edge e to get G, then v_m becomes adjacent to all the vertices of K_{d_l} in $con(G)$ which changes K_{d_l} to K_{d_l+1}. By the same argument, K_{d_m} becomes K_{d_m+1}. In $L(G)$ we have a new vertex which is adjacent to all the vertices of K_{d_l} and K_{d_m}. So K_{d_l} is changed to K_{d_l+1} and K_{d_m} changed to K_{d_m+1}. Also K_{d_l+1} and K_{d_m+1} have a common vertex. From Figure 2 it is clear that $con(G) \cong L(G)$

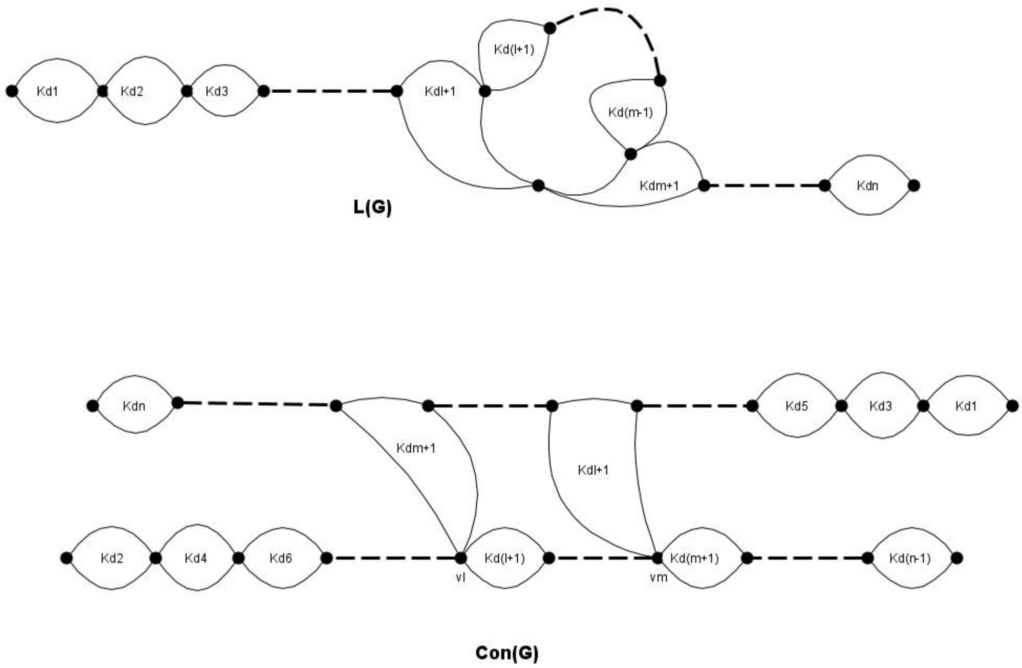

FIGURE 2

if and only if $V(K_{d_{m+1}}) = \{v_m\}$, $V(K_{d_{m+2}}) = V(K_{d_{m+3}}) = \cdots = \emptyset$ and $v_l = v_1$. Also in $con(G)$ the vertex v_l is common to three cliques, K_{d_m+1}, $K_{d_{l-1}}$ and $K_{d_{l+1}}$. But in $L(G)$ there does not exist any vertex common to three cliques and hence this case is not possible if l and m are even numbers. If l and m are odd numbers, by the above argument, v_l is common to three cliques, K_{d_m+1}, $K_{d_{l-1}}$ and $K_{d_{l+1}}$. But in this case when we consider, $v_l = v_1$, v_1 is common to K_{d_2} and K_{d_m+1} (Refer Figure 2).

Hence $L(G) \cong con(G)$ if and only if the sequence $d_1+1, d_2, d_3, \ldots, d_{m-1}, d_m+1$ is same as the sequence $d_1+1, d_3, d_5, , \ldots, d_{m-2}, d_m+1, d_2, d_4, \ldots, d_{m-1}$ in any order (Refer Figure 3).

Case 2: Let $e = u_l v_m$, where u_l is a pendant vertex adjacent to v_l. By Theorem 2.2, either l is even and m is odd or m is even and l is odd.

If l is even and m is odd, then the vertex v_l belongs to G_1 and both the vertices u_l and v_m belongs to G_2 in $con(G)$. When we add an edge joining u_l and v_m, u_l becomes adjacent to all the vertices of K_{d_m} in $con(G)$. So K_{d_m} is changed to K_{d_m+1}. Also, v_l becomes adjacent to v_m. In $L(G)$ we have a new vertex corresponding to e which is adjacent to exactly one

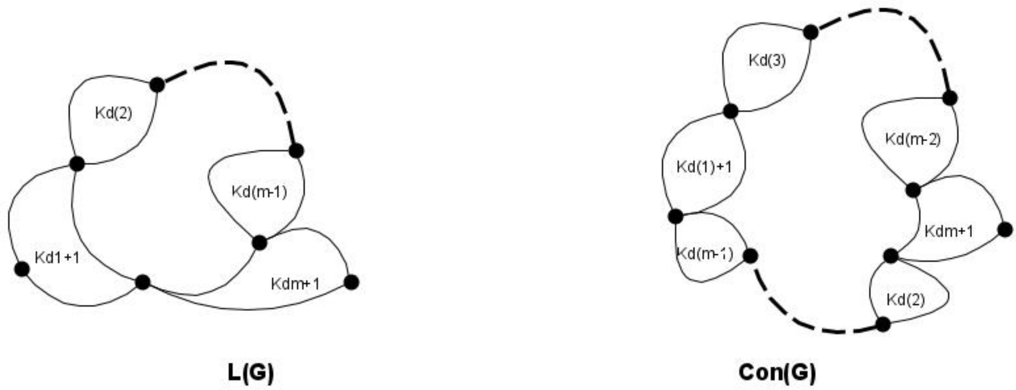

FIGURE 3

vertex of K_{d_l} and all the vertices of K_{d_m}. So K_{d_m} changed to K_{d_m+1} and there is an edge connecting one vertex of K_{d_l} with one vertex of K_{d_m+1}. Thus in $con(G)$ the vertex v_l is common to $K_{d_{l-1}}, K_{d_{l+1}}$ and K_2. But such a vertex do not exist in $L(G)$ implies $v_l = v_1$. which is a contradiction to our assumption that l is even.

If m is even and l is odd, we can prove as before that $v_l = v_1$. Also in $con(G)$, the vertex v_m is common to $K_{d_{m-1}}, K_{d_{m+1}}$ and K_2. But such vertex do not exists in $L(G)$ implies $V(K_{d_{m+1}}) = V(K_{d_{m+2}}) = \ldots = \emptyset$. As in Case 1 it is clear that $L(G) \cong con(G)$ if and only if the sequence $d_1, d_m+1, d_{m-2}, d_{m-4}, \ldots, d_2, 2, d_{m-1}, \ldots, d_3$ is a cyclic permutation of $d_1, d_2, d_3, \ldots, d_{m-1}, d_m+1, 2$.

Case 3: Let $e = u_l u_m$ and the non pendant vertices adjacent to u_l and u_m are v_l and v_m respectively. By Theorem 2.2, l and m are either both even or both odd. In this case there are three subcases.

Subcase 1: $v_l = v_m$.
In this case in $con(G)$, the vertex v_l, which is the common vertex of $K_{d_{l-1}}$ and $K_{d_{l+1}}$ becomes adjacent to both u_l and u_m (in K_{d_l}) in the other component of $con(T)$. On the other hand in $L(G)$, we have a new vertex which is adjacent to two vertices in K_{d_l}. If $L(G) \cong con(G)$, then $V(K_{d_{l+1}}) = \{v_l\}$ and $V(K_{d_{l+2}}) = V(K_{d_{l+3}}) = \cdots = \emptyset$. Then in $con(G)$ the vertex v_l becomes the common vertex of $K_{d_{l-1}}$ and K_3 and this K_3 and K_{d_l} have a common edge. Since such a vertex does not exist in $L(G)$, $V(K_{d_1}) = V(K_{d_2}) = \cdots = V(K_{d_{l-2}}) = \emptyset$ and $V(K_{d_{l-1}}) = \{v_l\}$. That is possible only when $v_l = v_1$. Hence we can conclude that $L(G) \cong con(G)$ if and only if G is a star graph together with one edge joining any two pendant vertices.
Subcase 2: l and m are even.
In $con(G)$, the vertex v_l, is the common vertex of $K_{d_{l-1}}$, $K_{d_{l+1}}$ and K_2. But in $L(G)$ there does not exist a vertex common to three cliques. So as in Case 2, we have to conclude that $v_l = v_1$, which is a contradiction to our assumption.
Subcase 3: l and m are odd.
In $con(G)$, the vertex v_l becomes adjacent to u_m, and v_m becomes adjacent to u_l. In $L(G)$, the vertex corresponding to the edge $u_l u_m$ is adjacent to one vertex in K_{d_l} (the vertex corresponding to the edge $v_l u_l$) and one vertex in K_{d_m}(the vertex corresponding to the edge $v_m u_m$). i.e., In $con(G)$ the vertex v_l is the common vertex of $K_{d_{l-1}}$,

$K_{d_{l+1}}$ and K_2 and the vertex v_m, is the common vertex of $K_{d_{m-1}}$, $K_{d_{m+1}}$ and K_2. But in $L(G)$ there does not exist a vertex common to three cliques. If $L(G) \cong con(G)$, then $V(K_{d_{m+1}}) = \{v_m\}$, $V(K_{d_{m+2}}) = V(K_{d_{m+3}}) = \cdots = \emptyset$ and $v_l = v_1$. Hence $L(G) \cong con(G)$ if and only if the sequence $d_2, d_4, d_6, \ldots, d_{m-1}, 2, d_1, d_3, \ldots, d_m, 2$ is a cyclic permutation of $d_1, d_2, d_3, \ldots, d_{m-1}, d_m, 2, 2$.

□

Conclusion

In this paper we have characterized the graphs for which the line graph and common neighborhood graph are isomorphic. Finding similar results for some other graph operators such as the Gallai graph, the anti-Gallai graph, block graph, total graph and middle graph is open.

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