CREAT. MATH. INFORM.
Volume **34** (2025), No. 1,
Pages 67-72

Online version at https://creative-mathematics.cunbm.utcluj.ro/
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2025.01.05

Timelike Bonnet surfaces on Lorentzian three-manifolds

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ABSTRACT. In this paper we generalized a result of Soley Ersoy and Kemal Eren [10] about Bonnet timelike surface in Minkowski 3-space. We give a necessary and sufficient condition for a surface M in a Lorentzian 3-space to be timelike Bonnet surface. Specifically, we give two theorems of classification of timelike Bonnet surface in a Lorentzian 3-space.

1. INTRODUCTION

The accurate knowledge of submanifolds of a given manifold is of great help while studying the features (geometric and topological properties) of the ambient manifold. The geometry of the Grassmanian of any manifold gives a total description of objects lying in the later. Therefore, it is interesting to get a natural and easy read of geometric properties of different level submanifolds of a given manifold. The study of surfaces on a manifold plays an important role in differential geometry. For more works about surfaces in three dimensional Lorentzian manifold, see [2, 9, 16, 17].
The Bonnet surfaces are surfaces which admit a one parameter family of isometric deformations preserving the principal curvatures. V. Lalan [11] was the first who used the term "*Bonnet surface*". In [1], Bonnet proved that any surface with constant mean curvature in Euclidean 3-space $\mathbb{E}$ (which is not totally umbilical) is a Bonnet surface. In [3], Cartan gave some detailed results for Bonnet surfaces and studied the existence and classification of Bonnet mates. He proved that in the Euclidean 3-space $\mathbb{E}$, any Bonnet surface is a *Weingarten surface*. If the ambient space of the surface is a complete simply connected Riemannian 3-manifold $\mathbb{R}^3(c)$ of constant curvature $c \geq 0$, W. Chen and H. Li showed that there exist always Bonnet surfaces which are not Weingarten surfaces [6] and in [7] they obtain the classification results of timelike Bonnet surfaces in 3-dimensional Minkowski space. Lawson in [12] extends Bonnet's results to any surface that has constant mean curvature in a Riemannian 3-manifold of constant curvature. Chern proved in [8] that differential forms can be used to characterize isometric deformations that maintain the principal curvatures of surfaces. In [13, 14, 15] I. Roussos has confirmed that helical surfaces may belong to a class of the Bonnet surface, and he considered the tangent developable surfaces as Bonnet surfaces. By using the method of Chern he also obtained a characterization for isometric deformation preserving the mean curvature. More detailed results concerned with these surfaces are presented in [5], where the authors showed that the mean curvature of W-surfaces satisfy an ordinary differential equation of third order and they found that surfaces with constant Gaussian curvature admitting such deformation should have zero Gaussian curvature. In [10] et al. obtained a necessary and sufficient condition for a timelike tangent developable surface to be a timelike Bonnet surface by the aid of this criterion. This is examined under the condition of the curvature and torsion of the base curve of the timelike developable surface being nonconstant. Moreover,

Received: 12.03.2024. In revised form: 25.05.2024. Accepted: 31.05.2024
2020 *Mathematics Subject Classification.* 53A10, 53C42, 53C50.
Key words and phrases. *Mean curvature, Principal curvature, Bonnet surface, Lorentzian space.*

they investigated the nontrivial isometry preserving the mean curvature for a timelike flat helicoidal surface by considering the curvature and torsion of the base curve of the timelike developable surface as being constant.
Motivated by the above works, in this papier we give some necessary conditions for a surface to be timelike Bonnet surface in a Lorentzian 3-space $\mathbb{L}^3$. We give a generalization of the works in [10]. The paper is organized as follow: in section 2, we recall some basic result and formulas about timelike surface without umbilic points in a three dimensional Lorentzian manifold. In section 3, we study geometric properties of the timelike Bonnet surfaces in $\mathbb{L}^3$. Specifically we give a necessary and sufficient condition for a surface M in a Lorentzian 3-space to be timelike Bonnet surface.

2. Some basic formulas

Let M be a local timelike surface of three dimensional Lorentzian manifold $\mathbb{L}^3$ defined by an embedding $x : M \to \mathbb{L}^3$. The curvature tensor of $\mathbb{L}^3$ is denote by R. The metric tensor of $\mathbb{L}^3$ is g. We will assume that the surface does not admit umbilic points and has a diagonalizable Weingarten operator A. Let (e_1, e_2) be an orthonormal basis on the local surface M which are eigenvectors of the shape operator A at any point of M. We will assume that e_1 is timelike and e_2 is spacelike. We put $e_3 = e_1 \times e_2$, where $\times$ is the vector product of $\mathbb{L}^3$.
Let $(\omega^1, \omega^2, \omega^3)$ be the dual basis of the local orthonormal frame (e_1, e_2, e_3) at the point $x \in M$. The Levi-Civita connection of $\mathbb{L}^3$ and the induced connection on M are both denote by ∇. We have the following equations:

$$\left\{ \begin{array}{rcl} dx &=& \omega^1 e_1 + \omega^2 e_2 \\ \nabla e_1 &=& \omega_1^2 e_2 + \omega_1^3 e_3 \\ \nabla e_2 &=& \omega_2^1 e_1 + \omega_2^3 e_3 \\ \nabla e_3 &=& \omega_1^3 e_1 - \omega_2^3 e_2 \end{array} \right. , \tag{2.1}$$

with $\omega_3^3 = \omega_1^1 = \omega_2^2 = 0$, $\omega_1^2 = \omega_2^1$, $\omega_1^3 = -\omega_3^1$ and $\omega_2^3 = -\omega_3^2$.
We define the shape operator $A : T_pM \to T_pM$ associated to the normal vector field e_3, by $A(e_1) = -ae_1$, $A(e_2) = -ce_2$. Thus this shape operator A is assumed to be diagonalizable. So we have

$$\omega_1^3 = a\omega^1; \quad \omega_2^3 = -c\omega^2. \tag{2.2}$$

The functions a and c are the principal curvature $(a \neq c)$. The mean curvature H and the Gauss curvature K of the surface M are given by

$$H = \frac{a+c}{2} \quad \text{and} \quad K = ac$$

respectively and we put $J = \frac{a-c}{2} > 0$, for more detail see [4, 7, 10].
We put

$$\omega_1^2 = h\omega^1 + k\omega^2,$$

where h and k are uniquely determined by the first structural equations of M given by

$$d\omega^1 = \omega^2 \wedge \omega_2^1; \quad d\omega^2 = \omega^1 \wedge \omega_1^2. \tag{2.3}$$

The Gauss and Codazzi equation of the surface M are given respectively by

$$d\omega_1^2 = -K\omega^1 \wedge \omega^2 + \psi_1^2, \tag{2.4}$$

with $\psi_1^2(X, Y) = g(R(X, Y)e_1, e_2)$ for all $X, Y \in TM$ and

$$\left\{ \begin{array}{l} d\omega_1^3 - \omega_1^2 \wedge \omega_2^3 = \psi_1^3 \\ d\omega_2^3 - \omega_2^1 \wedge \omega_1^3 = \psi_2^3 \end{array} \right. , \tag{2.5}$$

with $\psi_i^3(X,Y) = g(R(X,Y)e_i, e_3)$, $i = 1,2$, for all $X, Y \in TM$. We will put,

(2.6) $$\psi_1^3 = (c-a)\lambda_1^3\omega^2 \wedge \omega^1, \quad \psi_2^3 = (c-a)\lambda_2^3\omega^1 \wedge \omega^2,$$

Using the equations (2.3)-(2.6) and the fact that $\omega_1^2 = h\omega^1 + k\omega^2$ we have

$$\left[da + (a-c)(h+\lambda_1^3)\omega^2\right] \wedge \omega^1 = 0.$$

So there exist a function p' such that

$$da + (a-c)(h+\lambda_1^3)\omega^2 = p'\omega^1,$$

and we put $p' = p(c-a)$. So we have

$$da + (a-c)(h+\lambda_1^3)\omega^2 = p(c-a)\omega^1.$$

A same computation shows that there exist a function q such that

$$dc + (c-a)(k+\lambda_2^3)\omega^1 = (a-c)q\omega^2.$$

So in summary we get

$$\begin{cases} da + (a-c)(h+\lambda_1^2)\omega^2 = (c-a)p\omega^1 \\ dc + (c-a)(k+\lambda_2^3)\omega^1 = (a-c)q\omega^2 \end{cases}.$$

For more information and details see [10, 6].
Thus we have these following relations

(2.7) $$\begin{cases} \frac{da}{c-a} = p\omega^1 + (h+\lambda_1^3)\omega^2 \\ \frac{dc}{a-c} = q\omega^2 + (k+\lambda_2^3)\omega^1 \end{cases},$$

and

(2.8) $$2dH = (a-c)\left[\left((k+\lambda_2^3) - p\right)\omega^1 + \left(q - (h+\lambda_1^3)\right)\omega^2\right].$$

Now if we put

(2.9) $$\begin{cases} u = (k+\lambda_2^3) - p \\ v = q - (h+\lambda_1^3) \end{cases},$$

the equation (2.8) becomes

(2.10) $$2dH = (a-c)(u\omega^1 + v\omega^2).$$

We introduce the following 1-forms on the surface M

(2.11) $$\begin{cases} \theta^1 = u\omega^1 + v\omega^2 \\ \theta^2 = v\omega^1 + u\omega^2 \end{cases}.$$

The first fundamental form I of the surface M is given by

(2.12) $$I = (\omega^1)^2 - (\omega^2)^2$$

and then (by (2.10) and (2.12)), the gradient of the mean curvature H satisfied

(2.13) $$2gradH = (a-c)(ue_1 - ve_2).$$

We introduce again the following 1-forms

(2.14) $$\begin{cases} \alpha^1 = u\omega^1 - v\omega^2 \\ \alpha^2 = -v\omega^1 + u\omega^2 \end{cases}.$$

We define the $\star$ Hodge operator by

(2.15) $$\star\omega^1 = \omega^2, \quad \star\omega^2 = \omega^1, \quad (\star)^2 = 1,$$

and the connection form ω_1^2 and the forms θ^i, α^i, $i = 1,2$ in (2.11) and (2.14) satisfy

(2.16) $$\star\omega_1^2 = k\omega^1 + h\omega^2,$$

$$\left\{ \begin{array}{l} \star\theta^1 = \theta^2 \\ \star\theta^2 = \theta^1 \end{array} \right. \tag{2.17}$$

and

$$\left\{ \begin{array}{l} \star\alpha^1 = \alpha^2 \\ \star\alpha^2 = \alpha^1 \end{array} \right. . \tag{2.18}$$

From (2.7) and (2.9) we have

$$\left\{ \begin{array}{l} \frac{da}{a-c} = \left(u - (k+\lambda_2^3)\right)\omega^1 - (h+\lambda_1^3)\omega^2 \\ \frac{dc}{a-c} = (k+\lambda_2^3)\omega^1 + \left(v + (h+\lambda_1^3)\right)\omega^2 \end{array} \right. . \tag{2.19}$$

By using (2.19) and (2.16), we get

$$dln|a-c| = \alpha^1 - 2\star(\omega_2^1 + \beta), \tag{2.20}$$

where β statisfies

$$\left\{ \begin{array}{l} \beta = \lambda_1^3\omega^1 + \lambda_2^3\omega^2 \\ \star\beta = \lambda_2^3\omega^1 + \lambda_1^3\omega^2 \end{array} \right. . \tag{2.21}$$

3. Timelike Bonnet surface in $\mathbb{L}^3$

Here, we will study an analogue of the criterion given by [7, 10] for a timelike surface in a Lorentzian manifold $(\mathbb{L}, g)$ to be a Bonnet surface.
Let $\bar{M}$ be a new timelike surface in $\mathbb{L}^3$ with principal curvatures $\bar{a}$ and $\bar{c}$, which is an isometric deformation of M that preserve the first fundamental form and principal curvatures a and c of M (i.e an isometric $\varphi : M \to \bar{M}$ such that $\bar{a}\circ\varphi = a$ and $\bar{c}\circ\varphi = c$). Thus we have

$$\bar{a} = a, \qquad \bar{c} = c. \tag{3.22}$$

Let $\{\bar{\omega}^1, \bar{\omega}^2\}$ be coframe of the frame $\{\bar{e}_1, \bar{e}_2\}$ of $\bar{M}$. Since $\bar{M}$ and M have the same first fundamental form , we get

$$(\bar{\omega}^1)^2 - (\bar{\omega}^2)^2 = (\omega^1)^2 - (\omega^2)^2. \tag{3.23}$$

Then there exists some function φ on M such that

$$\left\{ \begin{array}{l} \bar{\omega}^1 = \cosh\varphi\omega^1 + \sinh\varphi\omega^2 \\ \bar{\omega}^2 = \sinh\varphi\omega^1 + \cosh\varphi\omega^2 \end{array} \right. . \tag{3.24}$$

By taking the exterior differentiation of (3.24) with the first structural equation, we can see easily that

$$\bar{\omega}_1^2 = \bar{\omega}_2^1 = \omega_1^2 - d\varphi.$$

The conditions $\bar{a} = a$, $\bar{c} = c$, the relations in (2.16), (2.21), with (2.20) give

$$\alpha^1 - 2\star(\omega_2^1 + \beta) = \bar{\alpha}^1 - 2\star(\bar{\omega}_2^1 + \bar{\beta}). \tag{3.25}$$

Applying the operator $\star$ in the equation (3.25) with equations (2.14-2.15) we get

$$\alpha^2 - 2(\omega_2^1 + \beta) = \bar{\alpha}^2 - 2(\bar{\omega}_2^1 + \bar{\beta}). \tag{3.26}$$

Since $\omega_1^2 - \bar{\omega}_2^1 = d\varphi$ and by the equation (3.25), the equation (3.26) becomes

$$\alpha^2 - \bar{\alpha}^2 = 2d\varphi + 2(\beta - \bar{\beta}). \tag{3.27}$$

The equation $-2dH = (\bar{a} - \bar{c})\bar{\theta}^1$ show that $\bar{\theta}^1 = \theta^1$. So $\bar{u}\bar{\omega}^1 + \bar{v}\bar{\omega}^2 = u\omega^1 + v\omega^2$; and combining with (3.24) give us

$$\left\{ \begin{array}{l} \bar{u} = u\cosh\varphi - v\sinh\varphi \\ \bar{v} = -u\sinh\varphi + v\cosh\varphi \end{array} \right. . \tag{3.28}$$

Now by using the equations (2.14) and (3.28) with the equation (3.24), an easy computation shows that

$$\bar{\alpha}^2 = \sinh 2\varphi \alpha^1 + \cosh 2\varphi \alpha^2. \tag{3.29}$$

We define $T = \coth \varphi$ and we use the relation in (3.27) to obtain

$$dT = T\alpha^1 + \alpha^2 + \frac{\beta - \bar{\beta}}{\sinh^2 \varphi}. \tag{3.30}$$

We can rewrite λ_1^3 and λ_2^3 given in (2.6) by the following expression

$$\begin{cases} \lambda_1^3 = \frac{2}{J} g(R(e_1, e_2)e_1, e_3) \\ \lambda_2^3 = -\frac{2}{J} g(R(e_1, e_2)e_2, e_3) \end{cases}. \tag{3.31}$$

By using the equation (3.24), we find

$$\begin{cases} \bar{e}_1 = \cosh \varphi e_1 - \sinh \varphi e_2 \\ \bar{e}_2 = -\sinh \varphi e_1 + \cosh \varphi e_2 \end{cases}, \tag{3.32}$$

and then we get

$$\bar{e}_1 \times \bar{e}_2 = e_1 \times e_2 = \bar{e}_3 = e_3. \tag{3.33}$$

Using the equations (3.31), (3.32) and (3.33) we get

$$\begin{cases} \bar{\lambda}_1^3 = \left(\cosh \varphi \lambda_1^3 + \sinh \varphi \lambda_2^3\right) \\ \bar{\lambda}_2^3 = \left(\sinh \varphi \lambda_1^3 + \cosh \varphi \lambda_2^3\right) \end{cases}, \tag{3.34}$$

and

$$\bar{\beta} = \cosh 2\varphi \beta + \sinh 2\varphi \star \beta. \tag{3.35}$$

One can see that β and $\bar{\beta}$ are related by

$$\frac{\beta - \bar{\beta}}{\sinh^2 \varphi} = -2(\beta + T(\star\beta)). \tag{3.36}$$

Now the equation (3.30) becomes

$$dT = T(\alpha^1 - 2 \star \beta) + (\alpha^2 - 2\beta). \tag{3.37}$$

Let now

$$\begin{cases} \gamma^1 = \alpha^1 - 2 \star \beta \\ \gamma^2 = \alpha^2 - 2\beta \end{cases}, \tag{3.38}$$

so the equation (3.37) become

$$dT = T\gamma^1 + \gamma^2. \tag{3.39}$$

Remark 3.1. *The equation (3.39) is a totally differential equation that the hyperbolic anlge T satisfies during the isometric deformation. In order to have non-trivial isometric deformation it is necessary and sufficient that the equation (3.39) to be completely integrable.*

Remark 3.2. *When H is constant then $u = v = 0$ and $\alpha^1 = \alpha^2 = 0$. Moreover if M has zero normal curvature (i.e $\lambda_1^3 = \lambda_2^3 = 0 \;\Leftrightarrow\; \beta = \star\beta = 0$), then $T = constant$ is solution of (3.39).*

So we have the following theorems:

Theorem 3.1. *Let M be timelike surface in $\mathbb{L}^3$ of constant mean curvature H with $H^2 > K$. If M has zero normal curvature then M has one parameter family of non trivial isometric deformations preserving the mean curvature; that is M is timelike Bonnet surface in $\mathbb{L}^3$.*

Theorem 3.2. *Let M be timelike surface in $\mathbb{L}^3$. If M has non-constant mean curvature H or has non-zero normal curvature then the exterior differentiation of (3.38) gives*

$$Td\gamma^1 + (d\gamma^2 - \gamma^1 \wedge \gamma^2) = 0; \tag{3.40}$$

So, the equation (3.39) is completely integrable iff

$$d\gamma^1 = 0 \quad ; \quad d\gamma^2 = \gamma^1 \wedge \gamma^2.$$

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