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Nonassociative van der Waerden rings

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ABSTRACT. The notion of a compact van der Waerden ring is modified for hereditarily linearly compact rings and a characterization of commutative hereditarily linearly compact metrizable van der Waerden rings is given. It is introduced the notion of an alternative(Jordan) van der Waerden ring similarly to the notion of a compact associative van der Waerden ring. We give a characterization of compact semisimple alternative (Jordan) van der Waerden rings similarly to the characterization of associative semisimple van der Waerden rings given by W.W. Comfort, D. Remus, H. Szambien, and M. Ursul.

1. INTRODUCTION

The notion of a van der Waerden associative ring has been introduced in [3]. The study of this class of compact rings has been continued in [10]. Topological rings close to the class of van der Waerden rings were studied in [11], [5]. We modify the notion of a van der Waerden ring for hereditarily linearly compact rings and derive a characterization of commutative semisimple hereditarily linearly compact metrizable van der Waerden rings.
Warner proved that compact ring topology of any associative semisimple compact ring is unique [16], or in other terms, that any associative semisimple compact ring is algebraically determined. We extend the Warner's result and the result of Comfort and Remus about semisimple van der Waerden rings [3] to semisimple alternative and Jordan rings.

2. NOTATION AND DEFINITIONS

Below $\omega = \{0, 1, 2, \ldots\}$ or the first infinite ordinal and $\mathbb{N} = \{1, 2, \ldots\}$. Rings are assumed to be unital, not necessarily associative. If S is a nonempty subset of a ring R, then the left annihilator of S is defined as follows: $Ann_l(S) = \{x \in R | xS = 0\}$. If R is an associative ring with identity then $U(R)$ stands for the group of units of R. An ideal I of a ring R is called cofinite if the factor ring R/I is finite. If R is a ring and $\mathfrak{m}$ a cardinal number then $\oplus_{\mathfrak{m}} R$ denotes the direct sum of $\mathfrak{m}$ copies of R. Topological rings are assumed to be Hausdorff. Ideal means a two sided ideal.
A topological associative ring $(R, \mathcal{T})$ is called to be *left linearly compact* [17],[7], [8] if it has a filter base of neighborhoods of zero consisting of left ideals and the intersection of every filter base consisting of closed cosets with respect to left ideals is nonempty. An associative topological ring is said to be *left hereditarily linearly compact* if every of its closed subring is left linearly compact.

Definition 2.1. *A topological ring is called linear if it has a filter base of neighbourhoods of zero consisting of ideals.*

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Definition 2.2. *A subring B of a topological ring A is called a retract if there exists a continuous homomorphism $f : A \to B$ such that $f(b) = b$ for each $b \in B$.*

3. Preliminaries

Recall the notion of a ultraproduct of rings (see [6], Chapter 7). Let $\{R_\alpha\}_{\alpha\in\Omega}$ be a family of rings and $\mathfrak{F}$ be a ultrafilter on Ω. Let $R = \prod_{\alpha\in\Omega} R_\alpha$ be the cartesian product of the rings $R_\alpha(\alpha \in \Omega)$.

The ideal $I_\mathfrak{F}$ is defined as follows: $(x_\alpha) \in I_\mathfrak{F}$ if and only if $\{\alpha \mid x_\alpha = 0_\alpha\} \in \mathfrak{F}$.

The factor ring $(\prod_{\alpha\in\Omega} R_\alpha)/I_\mathfrak{F}$ is called the *ultraproduct of the rings R_α*. Recall one result about the cardinality of a ultraproduct:

Theorem 3.1 ([4], Chapter 6, Corollary 6.8, p. 263; [9], Chapter IV, $ 8, Theorem 3, p. 220)**.** *The cardinality of any ultraproduct of a countable family of countable sets with respect to a non-principal ultrafilter is $2^{\aleph_0}$.*

Theorem 3.2. [13] *Every compact semisimple alternative (Jordan) ring has the form*

$$\prod_{\alpha\in\Omega} R_\alpha$$

where each R_α is a finite simple alternative (Jordan) ring.

4. Algebraically determined nonassociative rings

A compact ring $(R,\mathcal{U})$ is called *algebraically determined* if $\mathcal{U}' = \mathcal{U}$ for every compact ring topology $\mathcal{U}'$ on R. Algebraically determined rings were studied in [16].

Theorem 4.3. *Let $\{R_\alpha\}_{\alpha\in\Omega}$ be a family of finite not necessarily associative rings and $Ann_l(R_\alpha) = 0$ for each $\alpha \in \Omega$..*

Then $R = \prod_{\alpha\in\Omega} R_\alpha$ is an algebraically determined ring.

Proof. The ring R with the product topology $\mathcal{U}_0$ of discrete rings $R_\alpha(\alpha \in \Omega)$ is compact. We have $\{0_\alpha\} \times \prod_{\beta\neq\alpha} R_\beta = Ann_l(R_\alpha \times \prod_{\beta\neq\alpha}\{0_\beta\})$.
By continuity of the ring operations, $\{0_\alpha\} \times \prod_{\beta\neq\alpha} R_\beta$ is closed in any Hausdorff ring topology $\mathcal{U}$ on R. Since $\{0_\alpha\} \times \prod_{\beta\neq\alpha} R_\beta$ is a cofinite ideal of R it is open in $(R,\mathcal{U})$. Therefore $\mathcal{U}_0 \leq \mathcal{U}$, for any ring topology $\mathcal{U}$ on R. If $(R,\mathcal{U})$ is a compact ring (or a minimal topological ring), then $\mathcal{U} = \mathcal{U}_0$. □

Since any semisimple compact alternative (Jordan) ring is a product of finite rings with identity, it follows:

Corollary 4.1. *Any semisimple compact alternative (Jordan) ring is algebraically determined.*

Corollary 4.2. *For any finite not necessarily associative Boolean ring and for any cardinal number $\mathfrak{m}$ the compact ring $R^\mathfrak{m}$ is algebraically determined.*

Remark 4.1. *A nonassociative Boolean ring is constructed in* ([2], *Chapter II,$12, Example 8)*

Semisimple alternative ring means semisimple in the sense of Zhevlakov quasi-regular radical. The history of radicals in alternative and Jordan rings can be found in ([18], Chapters 10 and 14).

5. Semisimple commutative metrizable hereditarily linearly compact van der Waerden rings

Definition 5.3. *A left hereditarily linearly compact ring R is said to be a van der Waerden ring if every ring homomorphism $f : R \to R'$ of R in a left hereditarily linearly compact ring is continuous.*

Lemma 5.1. *Let $\{F_\alpha | \alpha \in \Omega\}$ be a family of fields and M be a maximal ideal of $\prod_{\alpha\in\Omega} R_\alpha$. Then either there exist a ultrafilter $\mathfrak{F}$ on Ω such that $M = I_{\mathfrak{F}}$ or $M = \{0_\beta\} \times \prod_{\alpha\neq\beta} F_\alpha$ for some $\beta \in \Omega$.*

Remark 5.2. *Lemma 5.1 is well-known. See, for instance, Lemma 7.3 in* [14].

Lemma 5.2. *A semisimple, metrizable, compact, commutative ring R is a van der Waerden ring in the class of compact rings if and only if R is a van der Waerden ring in the class of left hereditarily linearly compact rings.*

Proof. $\Rightarrow$: Let $R = \prod_{i\in\mathbb{N}} F_i$, where each F_i is a finite field and let $f : R \to R'$ be a homomorphism of R in a left hereditarily linearly compact ring R . Then R is a regular ring with identity, hence $f(R)$ is a commutative regular ring with identity. Consider the closure $\overline{f(R)}$ of $f(R)$. Since $\overline{f(R)}$ is a complete linear topological ring, it is an inverse limit, $\varprojlim(\overline{f(R)}/V)$, where V runs over all open ideals of $\overline{f(R)}$ and the projections are canonical. Every $\overline{f(R)}/V$ is regular, hence, semisimple. By [1], $\overline{f(R)}$ is a product of fields which are algebraic extensions of the finite fields. Let $\overline{f(R)} = \prod_{i=1}^{\infty} R_i$ or $\overline{f(R)} = \prod_{i=1}^{n} R_i$, where each R_i is an algebraic extension of a finite field.
Evidently, it suffices to show that if $pr_i \circ f : R \to R_i$ is continuous for each $i \in \mathbb{N}$. Clearly, $(pr_i \circ f)(R)$ is a finite or countable field.

If there exists $i \in \mathbb{N}$ such that $\ker(pr_i \circ f) = \{0_i\} \times \prod_{j\neq i} R_j$, then $pr_i \circ f$ is continuous. If $\ker(pr_i \circ f) = M = I_{\mathfrak{F}}$ for a ultrafilter $\mathfrak{F}$, this will contradict Lemma 5.1. It follows that $pr_i \circ f$ is always continuous.

$\Leftarrow$: Let $f : R \to R'$ be a homomorphism in a compact ring. Since R has identity, we can assume that R' has identity. Then R' will be hereditarily linearly compact, therefore f will be continuous. □

Theorem 5.4. *A commutative, semisimple, hereditarily linearly compact, metrizable ring R is a van der Waerden ring if and only if*

$R = A \times B$, a product of topological rings, where A is a commutative, compact semisimple van der Waerden ring and B is a commutative, metrizable hereditaily linearly compact ring which is a product of infinite algebraic extensions of finite fields.

Proof. $\Rightarrow$: Clearly, a direct summand of a van der Waerden ring is a van der Waerden ring. Represent R in the form $R = A \oplus B$, where A is a compact metrizable semisimple ring and B is a metrizable product of the infinite fields which are algebraic extensions of finite fields. By Lemma 5.2 A is a van der Waerden ring in the class of compact rings.

$\Leftarrow$: By Lemma 5.2, A is a van der Waerden ring in the class of hereditarily linearly compact rings. If B is discrete, the proof is finished. Assume that B is nondiscrete. Let $f : B \to R'$ be a homomorphism in a hereditarily linearly compact ring R'. Let $B = \prod_{\alpha\in\Omega} R_\alpha$. We can assume without loss of a generality that R' is an algebraic extension of a finite field and that f is surjective. If $\ker f = \{0_\beta\} \times \prod_{\alpha\neq\beta} R_\alpha$, the proof is finished. If $M = I_{\mathfrak{F}}$ for an ultrafilter $\mathfrak{F}$, then $| R' | \geq 2^{\aleph_0}$, a contradiction. Therefore f is continuous. □

6. Compact semisimple alternative and Jordan van der Waerden rings

Remark 6.3. *(folklore)Let k be a finite field considered as a topological ring with the discrete topology and $\mathfrak{m}$ an infinite cardinal number. Then $k^{\mathfrak{m}}$ contains a dense maximal cofinite ideal.*

Indeed, it suffices to take any maximal ideal M containing the ideal $\oplus_{\mathfrak{m}} k$.

Lemma 6.3. *If A is an arbitrary finite dimensional algebra with identity over a finite field F, then $A^{\mathfrak{m}}$, where $\mathfrak{m}$ is an infinite cardinal number, is not a van der Waerden ring.*

Proof. Since a retract of a van der Waerden ring is a van der Waerden ring it suffices to prove lemma for $\mathfrak{m} = \mathbb{N}$.

We will show that $A^{\mathbb{N}}$ has a cofinite dense maximal ideal. Let $B = A^{\mathbb{N}} = A_1 \times A_2 \times \cdots$. Denote by $pr_k : A^{\mathbb{N}} \to A_k$ the projection. Let $u_1, \ldots, u_n$ be a basis of A over F. Define $\tilde{u_i} \in A^{\mathbb{N}}, pr_k(\tilde{u_i}) = u_i$ for all $k \in \mathbb{N}$ and $n \in \mathbb{N}$.
We claim that $A^{\mathbb{N}} = F^{\mathbb{N}}\tilde{u_1} \oplus \cdots \oplus F^{\mathbb{N}}\tilde{u_n}$, a direct sum of submodules.
For, let $x = (x_1, x_2, \ldots) \in A^{\mathbb{N}}$. Then $x_i = \sum_{j=1}^{n} \alpha_j^{(i)} u_j$, where $\alpha_j^{(i)} \in F (i \in \mathbb{N})$, and $j \in \mathbb{N}$. Set $\tilde{\alpha_j} = (\alpha_j^{(i)}) \in A^{\mathbb{N}}$. If $k \in \mathbb{N}$, then $pr_k(\sum_{j=1}^{n} \tilde{\alpha_j}\tilde{u_j})$

$= \sum_{j=1}^{n} pr_k(\tilde{\alpha_j}) pr_k(\tilde{u_j}) = \sum_{j=1}^{n} \alpha_j^{(k)} u_j$
$= pr_k(x)$, hence, $x = \sum_{j=1}^{n} \tilde{\alpha_j}\tilde{u_j}$.

Let $\tilde{\alpha_1}\tilde{u_1} + \cdots + \tilde{\alpha_n}\tilde{u_n} = 0$. Assume, for instance, that $\tilde{\alpha_1} \neq 0$. Then $pr_k(\tilde{\alpha_1}) = \alpha_1^k \neq 0$ for some k. It follows that $pr_k(\tilde{\alpha_1}\tilde{u_1} + \cdots + \tilde{\alpha_n}\tilde{u_n}) = \alpha_1^k u_1 + \cdots + \alpha_n^k u_n = 0$ and $\alpha_1^k \neq 0$, a contradiction.

We have proved that $A^{\mathbb{N}} = F^{\mathbb{N}}\tilde{u_1} \oplus \cdots \oplus F^{\mathbb{N}}\tilde{u_n}$ is a direct sum of submodules. Since $F^{\mathbb{N}}$ is compact, $F^{\mathbb{N}}\tilde{u_i}$ also is compact. It follows that $A^{\mathbb{N}} = F^{\mathbb{N}}\tilde{u_1} \oplus \cdots \oplus F^{\mathbb{N}}\tilde{u_n}$ is a topological direct sum.

Let I is a cofinite dense ideal of $F^{\mathbb{N}}$. Then $I\tilde{u_1} \oplus \cdots \oplus I\tilde{u_n}$ will be cofinite dense ideal of $A^{\mathbb{N}}$.

The ring $A^{\mathbb{N}}$ belongs to the variety of rings generated by A. A variety of rings generated by a finite ring is locally finite ([9], Chapter 6). It follows that the factor ring $A^{\mathbb{N}}/(I\tilde{u_1} \oplus \cdots \oplus I\tilde{u_n})$ is finite.

By Remark 6.3 the ring $F^{\mathbb{N}}$ contains a dense cofinite ideal I. Then $I\tilde{u_1} \oplus \cdots \oplus I\tilde{u_n}$ will be a dense cofinite ideal of $A^{\mathbb{N}}$. It follows that $A^{\mathbb{N}}/(I\tilde{u_1} \oplus \cdots \oplus I\tilde{u_n})$ is finite, hence $A^{\mathbb{N}}$ is not a van der Waerden ring, a contradiction.

□

Theorem 6.5. *Let $R = \prod_{\alpha \in \Omega} R_\alpha$ be a semisimple alternative ring where each R_α is a finite simple associative or alternative ring. Then R is a van der Waerden ring if and only for each α the set of indexes β for which R_β is isomorphic to R_α is finite.*

Proof. $\Rightarrow$: Assume on the contrary that there exists $\alpha \in \Omega$ for which there exists an infinite number of indexes $\beta \in \Omega$ such that $R_\beta \cong R_\alpha$. Then $R = A^{\mathbb{N}} \times B$, a topological poduct, where A is an associative or alternative finite simple ring [13]. By Lemma 6.3, $A^{\mathbb{N}}$ is not a van der Waerden ring, a contradiction.

$\Leftarrow$: Represent R as a topological direct sum $R = A \times B$, where A is an associative ring and B is a topological product of Cayley-Dickson algebras over finite fields. By [3] A is a van der Waerden ring.

Since the topological product of a finite number of van der Waerden rings is a van der Waerden ring, it suffices to show that B is a van der Waerden ring.

Represent B in the form $B = B_1^{k_1} \times B_2^{k_2} \times \cdots$, where B_i are Cayley-Dickson algebras over finite fields, $\mid B_1 \mid < \mid B_2 \mid < \cdots$ and $k_1, k_2, \ldots$ are some natural numbers.

Each ring $B_i (i \in \mathbb{N})$ is an 8-dimensional algebra over its center which is a finite field F_i ([18], Corollary 1, p.151). Then $U(F_1) < U(F_2) < \cdots$.
The groups $U(F_i)$ are cyclic ([15], p.111). Denote by θ_i the generator of $U(F_i)(i \in \mathbb{N})$ and let $\lambda_i = \theta_i \times \theta_i \times \cdots \times \theta_i (k_i$ times). If l_i are orders of θ_i, then $l_1 < l_2 < \cdots$. Obviously, the order of λ_i is k_i. Consider the element $x = (\lambda_i)$.

Let I be a cofinite ideal of B. There exists $k \in \mathbb{N}$ such that $x^k - 1 \in I$. Let $i \in \mathbb{N}$ be such that $k < k_i$. Then $\lambda_j^k - e_j \neq 0$ for $j \geq i$, where e_j is the identity of B_j.

We notice that each element $\lambda_j^k - e_j$, where $j \geq i$ is in the center of $B_j^{k_j}$, hence it is invertible. Let
$y = (\lambda_i^k - e_i)^{k_i} \times (\lambda_{i+1}^k - e_{i+1})^{k_{i+1}} \times \cdots$. Clearly, y is an invertible element of $B_i^{k_i} \times B_{i+1}^{k_{i+1}} \times \cdots$. Let z be its inverse.

Then $(x^k - 1)z = e_i^{k_i} \times e_{i+1}^{k_{i+1}} \times \cdots \in I$. We have $B_i^{k_i} \times B_{i+1}^{k_{i+1}} \times \cdots = B(e_i^{k_i} \times e_{i+1}^{k_{i+1}} \times \cdots) \subseteq I$.

Since $B_i^{k_i} \times B_{i+1}^{k_{i+1}} \times \cdots$ is an open ideal of R, the ideal I is open. □

Using Lemma 6.3 we can prove similarly to Theorem 6.5

Theorem 6.6. *Let $R = \prod_{\alpha \in \Omega} R_\alpha$ be a semisimple Jordan ring, where each R_α is a finite simple Jordan ring. Then R is a van der Waerden ring if and only if for each α the set of indexes β for which R_β is isomorphic to R_α is finite.*

7. Examples

a) Cayley-Dickson algebras over finite fields $\mathbb{F}_{p^n}, p \neq 2, n \in \mathbb{N}$ are nonassociative simple finite alternative rings [18], $ 2.2, 2.3 and [12], chapter III, $ 5.

b) If p is a prime number $\neq 2$ then F_p with multiplication $x \circ y = \frac{1}{2}(xy + yx) = xy$ is a finite simple Jordan ring.

8. Conclusions

The aim of this article is generalization of some recent results of the theory of compact associative rings obtained by Remus D. and M.Ursul to compact alternative and Jordan rings.
It should be mentioned that methods used in the proofs are similar to methods used in the class of associative rings.

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