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Nonlinear elliptic problem involving natural growth term, L^1-data, variable exponent and Neumann boundary condition

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ABSTRACT. In this paper we study a class of multivalued Neumann boundary problem governed by the general $p(.)$-Leray-Lions type operator and involving a natural growth term and L^1 data. Using the technique of maximal monotone operator in Banach spaces and the approximation method via Yosida regularisation and penalizing term, we firstly prove the existence of at least one weak solution when the right hand side datum is bounded. Secondly, we deduce the existence of at least one renormalized solution when the right and side datum belongs in L^1. By choosing an appropriate test function, we end by establishing a relationship between renormalized solution and the entropy one.

1. INTRODUCTION

In the last ten years, the study of nonlinear partial differential equations in the framework of Sobolev spaces with variable exponent has undergone a considerable attention in the community of mathematic researchers. The main interest to such spaces rely on their efficient application in modelling the behaviour of various non-homogeneous materials in many fields such as physic, mechanical process, electro-rheological fluids, stationary thermo-rheological viscous flows of non-Newtonian fluids (see [4, 18, 23, 30, 31] for more details). They are also used in modelling the image processing ([15]).
In this paper we consider the following homogeneous nonlinear Neumann boundary value problem

$$(\mathcal{P}^g_{f,\beta})\begin{cases} \beta(u) - diva(x,u,\nabla u) + g(x,u,\nabla u) \ni f \ \text{ in } \ \Omega \\ \\ a(x,u,\nabla u)\cdot\nu = 0 \qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega, \end{cases}$$

where Ω is an open bounded domain of $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$, $f \in L^1(\Omega)$, $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ is a maximal monotone mapping such that $0 \in \beta(0)$, ν is the outer unit normal vector on $\partial\Omega$. The operator $A(u) = -diva(x,u,\nabla u)$ is called a $p(.)$-Leray-Lions type operator acting from $W^{1,p(.)}(\Omega)$ into its dual $(W^{1,p(.)}(\Omega))^*$.
Furthermore, g is a nonlinear term having natural growth (of order $p(.)$ where p is a function depending on x) with respect to gradient, which satisfies the sign-condition $g(x,s,\xi)s \geq 0$.
The aim of this paper is to show the existence of solutions of the problem $(\mathcal{P}^g_{f,\beta})$. Our approach is done in several steps. The first step consists to construct an approximating problem $(\mathcal{P}^{g_\epsilon}_{f,\beta_\epsilon})$ via Yosida regularization and penalization. In the second step, we use the technique of maximal monotone operator in Banach spaces to prove the existence of a sequence of solutions $(u_\epsilon)_{\epsilon>0}$ of the problem $(\mathcal{P}^{g_\epsilon}_{f,\beta_\epsilon})$. In the third step, we show that the

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sequence of solutions $(u_\epsilon)_{\epsilon>0}$ converges to a measurable function u which is a weak solution of the initial problem when the source term f is an L^∞-function. In the case where f belongs in $L^1(\Omega)$, we deduce the existence of a renormalized and/or an entropy solution. Let us stress that the Neumann boundary condition which appears on the boundary of problem $(\mathcal{P}^g_{f,\beta})$ requires to work in the space $W^{1,p(.)}(\Omega)$ instead of the common space $W_0^{1,p(.)}(\Omega)$. This situation creates several difficulties that we need to overcome in the present study. Going into details, the first difficulty which arises is that we cannot apply the Poincaré type inequality on the sequence of approximated solutions since they are expected in the space $W^{1,p(.)}(\Omega)$. Note also that this famous inequality is an important tool that one can use in homogeneous Dirichlet boundary condition case to achieve the coerciveness of the operator associated to the approximating problem. To overcome this difficulty, we add in the approximating problem a monotone function $\epsilon|u_\epsilon|^{p(x)-2}u_\epsilon$ (see Section 4 below). Taking into account the presence of this strong monotone perturbation, one can obtain the coerciveness of the associated operator to problem $(\mathcal{P}^{g_\epsilon}_{f,\beta_\epsilon})$ when the right hand side datum is an L^∞-function. The second difficulty we encounter is how to pass to the limit in the sequence of Yosida regularisation $(\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)))_{\epsilon>0}$ (see Section 4 below). To be able to pass to the limit as $\epsilon \to 0$ when the source term f belongs in L^∞, we establish an L^∞-estimate on the sequence $(\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)))_{\epsilon>0}$ which permits us to obtain its weak-* convergence to a function b in $L^\infty(\Omega)$. From this convergence, we deduce the existence of a sequence of functions $(b_n)_{n\in\mathbb{N}}$ which belong in L^1 and represents one component of the couple of solutions of the approximated problem $(\mathcal{P}^g_{f_n,\beta})$ in the case where the right hand side datum f belongs in L^1 in the initial problem (see Section 5). In order to pass to the limit as $n \to \infty$, we prove that the sequence $(b_n)_{n\in\mathbb{N}}$ is uniformly bounded and relatively weakly compact in $L^1(\Omega)$. This allows us to have its weak convergence to a function b in $L^1(\Omega)$.

In the literature, several authors have studied particular cases of the problem $(\mathcal{P}^g_{\beta,f})$. Letting the $p(.)$-Leray-Lions type operator be independent of u, β having a bounded domain and $g \equiv 0$, Ouaro and Ouédraogo (see [29]) have established the existence and uniqueness of an entropy solution of the following problem.

$$(P^0_{\beta,f})\begin{cases} \beta(u) - div\, a(x,\nabla u) \ni f & \text{in } \Omega \\ a(x,\nabla u).\eta = 0 & \text{on } \partial\Omega, \end{cases}$$

where f belongs in $L^1(\Omega)$. In [28], the authors have pushed the investigations of problem $(P^0_{\beta,f})$ by assuming that the right hand side data f is a diffuse measure. Other important works about inclusion differential problem can be found in [1, 2, 8]. In the framework of classic Sobolev spaces with constant exponent, many works in L^1-theory for nonlinear elliptic problem involving general p-Leray-Lions type operator and natural growth term have been analysed in [9, 10, 11, 12, 13, 24]. These works was further pushed forward into the framework of variable exponent [5, 17, 32]. As far as elliptic equations with natural growth terms and L^1-data under homogeneous Dirichlet boundary conditions are concerned, we refer the readers to [6, 7].

In a recent paper, Akdim et al. [3] have analysed the existence of solution of differential inclusion equation under homogeneous Dirichlet boundary condition in classic Sobolev space. The main interest in our work is that we are dealing with general nonlinear operators $-diva(x,u,\nabla u)$ and natural growth term under Neumann boundary condition in the context of Sobolev space with variable exponent.

Let us summarize the content of the paper. In Section 2, will present some definitions and properties of Sobolev spaces with variable exponents. In Section 3, we give our basic assumptions and recall a fundamental result. In Section 4, we prove the existence of weak solution when the right hand side datum $f \in L^{\infty}(\Omega)$ or $f \in L^{1}(\Omega)$. Finally, in Section 5, we prove the existence of renormalized solution when the right hand side datum belongs to $L^{1}(\Omega)$.

2. Preliminaries

We recall in what follows some definitions and basic properties of Lebesgue and Sobolev spaces with variable exponent.
Let Ω be a bounded domain in $\mathbb{R}^{N}$ $(N \geq 3)$ with smooth boundary $\partial\Omega$ and $p(.) : \overline{\Omega} \longrightarrow \mathbb{R}^{+}$ be a continuous function with

$$1 < p^{-} := \inf_{x\in\overline{\Omega}} p(x) \leq p^{+} := \sup_{x\in\overline{\Omega}} p(x) < \infty. \tag{2.1}$$

We denote

$$C_{+}(\overline{\Omega}) = \Big\{ p \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} p(x) > 1 \ \ a.e.\ x \in \Omega \Big\}.$$

For any $p \in C_{+}(\overline{\Omega})$, the variable exponent Lebesgue space is defined by

$$L^{p(.)}(\Omega) := \Big\{ u : \Omega \to \mathbb{R} \text{ measurable} : \int_{\Omega} |u|^{p(x)} dx < \infty \Big\},$$

endowed with the so-called Luxembourg norm

$$\|u\|_{p(.)} := \inf \Big\{ \lambda > 0 : \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \Big\}.$$

The $p(.)$-modular of the $L^{p(.)}(\Omega)$ space is the mapping $\rho_{p(.)} : L^{p(.)}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\rho_{p(.)}(u) := \int_{\Omega} |u|^{p(x)} dx.$$

For any $u \in L^{p(.)}(\Omega)$, the following inequality (see [21],[22]) will be used later.

$$\min \Big\{ \|u\|_{p(.)}^{p^{-}};\ \|u\|_{p(.)}^{p^{+}} \Big\} \leq \rho_{p(.)}(u) \leq \max \Big\{ \|u\|_{p(.)}^{p^{-}};\ \|u\|_{p(.)}^{p^{+}} \Big\}. \tag{2.2}$$

For any $u \in L^{p(.)}(\Omega)$ and $v \in L^{q(.)}(\Omega)$, with $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$ in Ω, we have the Hölder type inequality (see [26]).

$$\left| \int_{\Omega} uv dx \right| \leq \left(\frac{1}{p^{-}} + \frac{1}{q^{-}} \right) \|u\|_{p(.)} \|v\|_{q(.)}. \tag{2.3}$$

If Ω is bounded and $p,\ q \in C_{+}(\overline{\Omega})$ such that $p(x) \leq q(x)$ for any $x \in \Omega$, then the embedding $L^{q(.)}(\Omega) \hookrightarrow L^{p(.)}(\Omega)$ is continuous (see [26], Theorem 2.8).

Proposition 2.1. [26] *For $u_n, u \in L^{p(.)}(\Omega)$ and $p_{+} < \infty$, the following assertion hold.*

(i) $\|u\|_{p(.)} < 1$ *(resp,* $= 1,\ > 1$*) if and only if* $\rho_{p(.)}(u) < 1$ *(resp,* $= 1,\ > 1$*);*

(ii) $\|u\|_{p(.)} > 1$ *imply* $\|u\|_{p(.)}^{p_{-}} \leq \rho_{p(.)}(u) \leq \|u\|_{p(.)}^{p_{+}}$*, and* $\|u\|_{p(.)} < 1$ *imply* $\|u\|_{p(.)}^{p_{+}} \leq \rho_{p(.)}(u) \leq \|u\|_{p(.)}^{p_{-}}$*;*

(iii) $\|u_n\|_{p(.)} \to 0$ *if and only if* $\rho_{p(.)}(u_n) \to 0$*, and* $\|u_n\|_{p(.)} \to \infty$ *if and only* $\rho_{p(.)}(u_n) \to \infty$*.*

Now, we define the variable exponent Sobolev space by

$$W^{1,p(.)}(\Omega) := \Big\{ u \in L^{p(.)}(\Omega) :\ |\nabla u| \in L^{p(.)}(\Omega) \Big\},$$

with the norm

$$||u||_{1,p(.)} = \|u\|_{p(.)} + \|\nabla u\|_{p(.)}.$$

We denote by $W_0^{1,p(.)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(.)}(\Omega)$, and we define the Sobolev exponent by $p^*(.) = \frac{Np(.)}{N-p(.)}$ if $p(.) < N$ and $p^*(.) = \infty$ if $p(.) \geq N$.

Lemma 2.1. [6] *For $1 < p(.) < \infty$ and $u, u_n \in L^{p(.)}(\Omega)$ such that $\|u_n\|_{p(.)} \leq C$, if $u_n(.) \to u(.)$ a.e. in Ω, then $u_n \rightharpoonup u$ in $L^{p(.)}(\Omega)$.*

Theorem 2.1. [22, 25]

(i) Assuming $1 < p_- \leq p^+ < \infty$, the spaces $L^{p(.)}(\Omega)$ and $W_0^{1,p(.)}(\Omega)$ are separable and reflexive Banach spaces.
(ii) If $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \Omega$, then the embedding $W_0^{1,p(.)}(\Omega) \hookrightarrow\hookrightarrow L^{q(.)}(\Omega)$ is continuous and compact.
(iii) Poincaré inequality : there exists a constant $C > 0$, such that

$$\|u\|_{p(.)} \leq C\|\nabla u\|_{p(.)}, \ \forall u \in W_0^{1,p(.)}(\Omega).$$

(iv) Sobolev-Poincaré inequality : there exists a constant $C > 0$, such that

$$\|u\|_{p^*(.)} \leq C\|\nabla u\|_{p(.)}, \ \forall u \in W_0^{1,p(.)}(\Omega).$$

Remark 2.1. *By (iii) of Theorem 2.1, we deduce that $\|\nabla u\|_{p(.)}$ and $\|u\|_{1,p(.)}$ are equivalent norms in $W_0^{1,p(.)}(\Omega)$.*

Definition 2.1. [19] *For any $x \in \Omega$ such that $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, we denote the dual of the Sobolev space $W_0^{1,p(.)}(\Omega)$ by $W^{-1,p'(.)}(\Omega)$, and for each $F \in W^{-1,p'(.)}(\Omega)$, there exists $f_0, f_1, ..., f_N \in L^{p'(.)}(\Omega)$ such that $F = f_0 + \sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i}$. Moreover, for all $u \in W_0^{1,p(.)}(\Omega)$, one has*

$$\langle F, u \rangle = \int_\Omega f_0 u dx - \sum_{i=1}^{N} \int_\Omega f_i \frac{\partial u}{\partial x_i} dx,$$

and we define a norm on the dual space by

$$\|F\|_{-1,p'(.)} \simeq \sum_{i=0}^{N} \|f_i\|_{p'(.)}.$$

Finally, we use throughout the paper, the truncation function T_k, $(k > 0)$ defined by

$$T_k(s) = \max\{-k, \min\{k; s\}\}. \tag{2.4}$$

It is clear that $\lim_{k\to\infty} T_k(s) = s$ and $|T_k(s)| = \min\{|s|; k\}$.
$\mathcal{T}^{1,p(.)}(\Omega) := \{u : \Omega \to \mathbb{R}$ measurable function such that $T_k(u) \in W^{1,p(.)}(\Omega)\}$.
Let us introduce some functions that will be frequently used in this paper.
For $r \in \mathbb{R}$, let $r^+ := \max(r, 0)$ and sign_0^+ be the function defined by

$$sign_0^+(r) = \begin{cases} 1 \text{ if } r > 0, \\ 0 \text{ if } r \leq 0. \end{cases}$$

For $\delta > 0$, we define $H_\delta^+ : \mathbb{R} \to \mathbb{R}$ by

$$H_\delta^+(r) = \begin{cases} 1 \ \text{ if } r > \delta \\ \dfrac{r}{\delta} \ \text{ if } 0 \leq r \leq \delta \\ 0 \ \text{ if } r < 0. \end{cases}$$

Clearly, H_δ^+ is an approximation of $sign_0^+$.

Definition 2.2. [14] *Let $(\Omega, \mathcal{M}, \mu)$ be a finite measure space. A bounded subset $\mathcal{F}$ of $L^1(\mu)$ is called uniformly integrable if for every $\epsilon > 0$ and $f \in \mathcal{F}$, one has*

- *there exists $\delta > 0$ such that $\displaystyle\int_A |f| d\mu < \epsilon$ if $\mu(A) < \delta$*

and

- *there exists $\omega \subset \Omega$ measurable with $|\omega| < \infty$ such that $\displaystyle\int_{\Omega \setminus \omega} |f| d\mu < \epsilon$.*

Proposition 2.2. [16] *Let $(\Omega, \mathcal{M}, \mu)$ be a finite measure space. A subset $\mathcal{F}$ of $L^1(\mu)$ is called uniformly integrable if one has*

$$\lim_{a \to \infty} \left(\sup_{f \in \mathcal{F}} \int_{\{|f| \geq a\}} |f| d\mu \right) = 0.$$

Proposition 2.3. [16] *If the measure μ is bounded, any subset $\mathcal{F}$ in L^1, bounded in L^∞, is uniformly integrable.*

Proposition 2.4. [16] *Let $\mathcal{F}$ be a subset of $L^1(\mu)$. If there exists a positive function $f \in L^1$ such that $|g| \leq f$ for any $g \in \mathcal{F}$, then, $\mathcal{F}$ is uniformly integrable.*

Theorem 2.2. [20] *(Dunford). A subset of $L^1(\mu)$ is relatively weakly compact if and only if it is bounded and uniformly integrable.*

Lemma 2.2. [6] *Let $u \in L^{r(\cdot)}(\Omega)$ and $u_n \in L^{r(\cdot)}(\Omega)$ such that $\|u_n\|_{L^{r(\cdot)}(\Omega)} \leq C$ for $1 < r(x) < \infty$. If $u_n \rightharpoonup u$ a.e. in Ω, then $u_n \rightharpoonup u$ in $L^{r(\cdot)}(\Omega)$.*

3. Assumption and fundamental result

The data involved in our study are subject to the following conditions.
$a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying the following assumptions for almost every $x \in \Omega$ and for all $\xi, \eta \in \mathbb{R}^N$, $s \in \mathbb{R}$.

$$a(x, s, \xi).\xi \geq \lambda |\xi|^{p(x)}; \tag{3.5}$$

$$|a(x, s, \xi)| \leq \beta(k(x) + |s|^{p(x)-1} + |\xi|^{p(x)-1}); \tag{3.6}$$

$$(a(x, s, \xi) - a(x, s, \eta))(\xi - \eta) > 0 \text{ if } \xi \neq \eta; \tag{3.7}$$

where λ, β are two positive constants and $k(.)$ is a given nonnegative function in $L^{p'(\cdot)}(\Omega)$.

$$a(x, s, 0) = 0. \tag{3.8}$$

Furthermore, $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function such that for almost every $x \in \Omega$, $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$,

$$g(x, s, \xi)s \geq 0; \tag{3.9}$$

$$|g(x, s, \xi)| \leq b(|s|)(c(x) + |\xi|^{p(x)}); \tag{3.10}$$

where $b:\mathbb{R}^+\to\mathbb{R}^+$ is a continuous increasing function, $c(.)$ a given nonnegative function in $L^1(\Omega)$.
In the sequel, the following lemma which proof follows the same lines as in [6] will be useful.

Lemma 3.3. [6] *Assuming that* (3.5)-(3.7) *hold and* $(u_n)_{n\in\mathbb{N}}$ *is a sequence in* $W^{1,p(.)}(\Omega)$ *such that* $u_n \rightharpoonup u$ *in* $W^{1,p(.)}(\Omega)$ *and*

$$\int_\Omega \Big[a(x,u_n,\nabla u_n)-a(x,u_n,\nabla u)\Big]\nabla(u_n-u)dx\to 0. \tag{3.11}$$

Then, $u_n\to u$ *in* $W^{1,p(.)}(\Omega)$.

4. EXITENCE OF WEAK SOLUTION

This part contains the first main results of the problem $(\mathcal{P}^g_{f,g})$ using the notion of weak solution.

Theorem 4.3. *For* $f\in L^\infty(\Omega)$ *or in* $L^1(\Omega)$, *there exists at least one weak solution* $(u,b)\in\mathcal{T}^{1,p(.)}(\Omega)\times L^1(\Omega)$ *of problem* $(\mathcal{P}^g_{f,\beta})$ *in the sense that* $b(x)\in\beta(u(x))$ *a.e. in* Ω, $g(x,u,\nabla u)\in L^1(\Omega)$ *and*

$$\int_\Omega b\varphi dx+\int_\Omega a(x,u,\nabla u)\nabla\varphi dx+\int_\Omega g(x,u,\nabla u)\varphi dx=\int_\Omega f\varphi dx, \tag{4.12}$$

for any $\varphi\in W^{1,p(.)}(\Omega)\cap L^\infty(\Omega)$.

Proof. In this section we prove into tree steps, the existence of at least one weak solution of the problem $(\mathcal{P}^g_{f,\beta})$ when the right and side datum f belongs to $L^\infty(\Omega)$.
4.1. **Proof of the case** $f\in L^\infty(\Omega)$.
Step 1. Approximated problem
We consider the sequence of approximate problem

$$(\mathcal{P}^{g_\epsilon}_{f,\beta_\epsilon})\begin{cases}\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))-diva(x,u_\epsilon,\nabla u_\epsilon)+g_\epsilon(x,u_\epsilon,\nabla u_\epsilon)+\epsilon|u_\epsilon|^{p(x)-2}u_\epsilon=f \text{ in } \Omega\\ a(x,u_\epsilon,\nabla u_\epsilon)\cdot\eta=0 \qquad \text{on } \partial\Omega,\end{cases}$$

where $\beta_\epsilon:\mathbb{R}\to\mathbb{R}$ is the Yosida approximation of β and $g_\epsilon(x,s,\xi)=\dfrac{g(x,s,\xi)}{1+\epsilon|g(x,s,\xi)|}$, for any $\epsilon\in(0,1]$.
For all $u\in W^{1,p(.)}(\Omega)$, remark that

$$\langle\beta_\epsilon(u),u\rangle\geq 0,\ |\beta_\epsilon(u)|\leq\frac{1}{\epsilon}|u| \text{ and } \lim_{\epsilon\to 0}\beta_\epsilon(u)=\beta(u).$$

One also has

$$g_\epsilon(x,s,\xi)s\geq 0,\ \ |g_\epsilon(x,s,\xi)|\leq|g(x,s,\xi)|\ ,\ |g_\epsilon(x,s,\xi)|\leq\frac{1}{\epsilon}$$

and

$$|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))|\leq\frac{1}{\epsilon^2}.$$

Theorem 4.4. *For any* $f\in(W^{1,p(.)}(\Omega))^*$, *the problem* $(\mathcal{P}^{g_\epsilon}_{f,\beta_\epsilon})$ *admits at least one weak solution* $u_\epsilon\in W^{1,p(.)}(\Omega)$. *Namely,*

$$\int_\Omega\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))\varphi dx+\int_\Omega a(x,u_\epsilon,\nabla u_\epsilon)\nabla\varphi dx+\int_\Omega g_\epsilon(x,u_\epsilon,\nabla u_\epsilon)\varphi dx+\epsilon\int_\Omega|u_\epsilon|^{p(x)-2}u_\epsilon\varphi dx$$

$$= \int_\Omega f\varphi dx, \tag{4.13}$$

for all $\varphi \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$.

Proof. Let us define the operator $A_\epsilon : W^{1,p(.)}(\Omega) \to (W^{1,p(.)}(\Omega))^*$ as follows.
$\forall\, u, \varphi \in W^{1,p(.)}(\Omega)$,

$$\langle A_\epsilon(u), \varphi\rangle = \langle Au, \varphi\rangle + \int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u))\varphi dx + \int_\Omega g_\epsilon(x, u, \nabla u)\varphi dx + \epsilon \int_\Omega |u_\epsilon|^{p(x)-2}u_\epsilon dx,$$

where $\langle Au, \varphi\rangle = \displaystyle\int_\Omega a(x, u, \nabla u)\nabla\varphi dx$.

Lemma 4.4. *The operator* A_ϵ *is pseudo-monotone and bounded. Moreover,* A_ϵ *is coercive in the following sense.*

$$\frac{\langle A_\epsilon(u), u\rangle}{\|u\|_{1,p(.)}} \to \infty \text{ as } \|u\|_{1,p(.)} \to \infty.$$

Proof. There exists a constant $C > 0$ such that (see [32])

$$\left| \int_\Omega g_\epsilon(x, u, \nabla u)\varphi dx \right| \le C\|\varphi\|_{1,p(.)}. \tag{4.14}$$

Since $\beta_\epsilon \circ T_{\frac{1}{\epsilon}}$ is bounded in $L^{p'(.)}(\Omega)$, there exists a constant $C' > 0$ such that, by using Hölder type inequality, one gets

$$\left| \int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u))\varphi dx \right| \le \int_\Omega |\beta_\epsilon(T_{\frac{1}{\epsilon}}(u))\varphi| dx \le \left(\frac{1}{p^-} + \frac{1}{(p^-)'}\right)\|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u))\|_{p'(.)}\|\varphi\|_{p(.)}$$

$$\le C'\|\varphi\|_{1,p(.)}. \tag{4.15}$$

By using again Hölder type inequality, one has

$$\begin{aligned}
\left|\epsilon \int_\Omega |u_\epsilon|^{p(x)-2}u_\epsilon dx\right| &\le \epsilon \int_\Omega |u_\epsilon|^{p(x)-1}|\varphi| dx \\
&\le \epsilon\left(\frac{1}{p_-} + \frac{1}{(p_-)'}\right)\||u_\epsilon|^{p(x)-1}\|_{p'(.)}\|\varphi\|_{p(.)} \\
&\le \epsilon\left(\frac{1}{p_-} + \frac{1}{(p_-)'}\right)\||u_\epsilon|^{p(x)-1}\|_{p'(.)}(\|\varphi\|_{p(.)} + \|\nabla\varphi\|_{p(.)}) \le C\|\varphi\|_{1,p(.)},
\end{aligned}$$

where $C = \epsilon\left(\frac{1}{p_-} + \frac{1}{(p_-)'}\right)\||u_\epsilon|^{p(x)-1}\|_{p'(.)}$.
From the Hölder type inequality and the growth condition (3.10), we can prove that A is bounded. Then, we deduce from (3.6), (4.14) and (4.15) that A_ϵ is bounded.
To prove the coercivity of A_ϵ, we set

$$\alpha = \begin{cases} p_+ \text{ if } \|u\|_{1,p(.)} \le 1, \\ p_- \text{ if } \|u\|_{1,p(.)} > 1; \end{cases}$$

then, for all $u \in W^{1,p(.)}(\Omega)$, one has

$$\begin{aligned}\frac{\langle A_\epsilon(u), u\rangle}{\|u\|_{1,p(.)}} &= \frac{\langle Au, u\rangle + \int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u))udx + \int_\Omega g_\epsilon(x,u,\nabla u)udx + \epsilon\int_\Omega |u|^{p(x)-2}udx}{\|u\|_{1,p(.)}} \\ &\geq \frac{\int_\Omega a(x,u,\nabla u)\nabla udx + \epsilon\int_\Omega |u|^{p(x)-2}udx}{\|u\|_{1,p(.)}} \qquad \text{(by neglecting positive terms)} \\ &\geq \frac{\lambda\int_\Omega |\nabla u|^{p(x)}dx + \epsilon\int_\Omega |u|^{p(x)-2}udx}{\|u\|_{1,p(.)}} \\ &\geq \frac{\min(\epsilon,\lambda)\left(\int_\Omega |\nabla u|^{p(x)}dx + \int_\Omega |u|^{p(x)}dx\right)}{\|u\|_{1,p(.)}} \geq \frac{\min(\epsilon,\lambda)\rho_{1,p(.)}(u)}{\|u\|_{1,p(.)}} \\ &\geq \min(\epsilon,\lambda)\frac{\|u\|^\alpha_{1,p(.)}}{\|u\|_{1,p(.)}} \geq \min(\epsilon,\lambda)\|u\|^{\alpha-1}_{1,p(.)} \to \infty \text{ as } \|u\|_{1,p(.)}\to\infty \text{ (since } 1<p_-\leq p_+).\end{aligned}$$

Therefore A_ϵ is coercive.
Now it remains to establish that A_ϵ is pseudo-monotone. Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $W^{1,p(.)}(\Omega)$ such that

$$\left\{\begin{array}{l} u_k \rightharpoonup u \text{ in } W^{1,p(.)}(\Omega), \\ \\ A_\epsilon u_k \rightharpoonup \chi \text{ in } (W^{1,p(.)}(\Omega))^*, \\ \\ \lim_{k\to\infty}\sup\langle A_\epsilon u_k, u_k\rangle \leq \langle \chi_\epsilon, u\rangle. \end{array}\right. \tag{4.16}$$

Let us show that

$$\langle A_\epsilon u_k, u_k\rangle \longrightarrow \langle \chi, u\rangle \text{ as } k\longrightarrow\infty, \text{ where } \chi = A_\epsilon u.$$

According to the compact embedding $W^{1,p(.)}(\Omega) \hookrightarrow\hookrightarrow L^{p(.)}(\Omega)$, there exists a subsequence still denoted $(u_k)_{k\in\mathcal{N}}$ such that $u_k \to u$ in $L^{p(.)}(\Omega)$ as $k\to\infty$.
As $(u_k)_{k\in\mathbb{N}}$ is a bounded sequence in $W^{1,p(.)}(\Omega)$, using the growth condition it follows that $(a(x,u_k,\nabla u_k))_{k\in\mathbb{N}}$ is bounded in $(L^{p'(.)}(\Omega))^N$. Then, there exists a function $\varphi \in (L^{p'(.)}(\Omega))^N$ such that

$$a(x,u_k,\nabla u_k) \rightharpoonup \varphi \text{ in } (L^{p'(.)}(\Omega))^N \text{ as } k\to\infty. \tag{4.17}$$

Since $(g_\epsilon(x,u_k,\nabla u_k))_{k\in\mathbb{N}}$ is bounded in $(L^{p'(.)}(\Omega))^N$, we similarly deduce the existence of a function $\psi_\epsilon \in (L^{p'(.)}(\Omega))^N$ such that

$$g_\epsilon(x,u_k,\nabla u_k) \rightharpoonup \psi_\epsilon \text{ in } (L^{p'(.)}(\Omega))^N \text{ as } k\to\infty. \tag{4.18}$$

In view of the inequality $|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_k))| \leq \frac{1}{\epsilon^2}$ and the convergence $u_k \longrightarrow u$ a.e. in Ω as $k\to\infty$, one deduces from the Lebesgue dominated convergence theorem that

$$\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_k)) \longrightarrow \beta_\epsilon(T_{\frac{1}{\epsilon}}(u)) \text{ in } L^{(p_-)'}(\Omega) \text{ as } k\to\infty. \tag{4.19}$$

On the other hand, the generalized Lebesgue convergence theorem implies the following.

$$\epsilon|u_k|^{p(x)-2}u_k \longrightarrow \epsilon|u|^{p(x)-2}u \text{ strongly in } L^{p'(.)}(\Omega) \text{ as } k\to\infty. \tag{4.20}$$

Thus, for any $v \in W^{1,p(\cdot)}(\Omega)$,

$$\begin{aligned}\langle \chi_\epsilon, v\rangle &= \lim_{k\to\infty}\langle A_\epsilon u_k, v\rangle \\ &= \lim_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla v dx + \lim_{k\to\infty}\int_\Omega g_\epsilon(x,u_k,\nabla u_k) v dx \\ &+ \lim_{k\to\infty}\int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_k)) v dx + \epsilon\lim_{k\to\infty}\int_\Omega |u_k|^{p(x)-2}u_k dx \\ &= \int_\Omega \varphi\nabla v dx + \int_\Omega \psi_\epsilon v dx + \int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u)) v dx + \epsilon\int_\Omega |u|^{p(x)-2}u dx. \end{aligned} \tag{4.21}$$

Having in mind (4.16) and (4.21), one obtains

$$\begin{aligned}\lim_{k\to\infty}\sup\langle A_\epsilon u_k, u_k\rangle &= \lim_{k\to\infty}\sup\Bigg(\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k dx + \epsilon\int_\Omega |u_k|^{p(x)}dx \\ &+ \int_\Omega g_\epsilon(x,u_k,\nabla u_k)u_k dx + \int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_k))u_k dx\Bigg) \\ &\le \int_\Omega \varphi\nabla u dx + \int_\Omega \psi_\epsilon u dx + \int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u)) u dx \\ &+\epsilon\int_\Omega |u|^{p(x)}dx. \end{aligned} \tag{4.22}$$

Using respectively (4.18), (4.19) and (4.20) when $k \to \infty$, one has

$$\int_\Omega g_\epsilon(x,u_k,\nabla u_k)u_k dx \longrightarrow \int_\Omega \psi_\epsilon u dx, \tag{4.23}$$

$$\int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_k))u_k dx \longrightarrow \int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u))u dx \tag{4.24}$$

and

$$\epsilon\int_\Omega |u_k|^{p(x)-2}u_k \longrightarrow \epsilon\int_\Omega |u|^{p(x)-2}u dx. \tag{4.25}$$

It follows that

$$\lim_{k\to\infty}\sup\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k dx \le \int_\Omega \varphi\nabla u dx. \tag{4.26}$$

Thanks to (3.7), one has

$$\int_\Omega (a(x,u_k,\nabla u_k) - a(x,u_k,\nabla u))(\nabla u_k - \nabla u)dx \ge 0,$$

then,

$$\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k dx \ge \int_\Omega a(x,u_k,\nabla u_k)\nabla u dx + \int_\Omega a(x,u_k,\nabla u)(\nabla u_k - \nabla u)dx.$$

Since $\nabla u_k \rightharpoonup \nabla u$ in $L^{p(\cdot)}(\Omega)$, using (4.17) one obtains

$$\lim_{k\to\infty}\inf\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k dx \ge \int_\Omega \varphi\nabla u dx.$$

From (4.26), one can deduce that

$$\lim_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k dx = \int_\Omega \varphi\nabla u dx. \tag{4.27}$$

Therefore, combining (4.23), (4.24), (4.25) and (4.27), one obtains

$$\langle A_\epsilon u_k, u_k\rangle \to \langle \chi_\epsilon, u\rangle \text{ as } k\to\infty.$$

It remain to prove that $a(x,u_k,\nabla u_k) \rightharpoonup a(x,u,\nabla u)$ in $(L^{p'(.)}(\Omega))^N$ and

$$g_\epsilon(x,u_k,\nabla u_k) \rightharpoonup g_\epsilon(x,u,\nabla u) \text{ in } L^{p'(.)}(\Omega) \text{ as } k\to\infty.$$

From (4.27), we can prove that

$$\lim_{k\to\infty}\int_\Omega (a(x,u_k,\nabla u_k)-a(x,u_k,\nabla u))(\nabla u_k-\nabla u)dx=0.$$

According to Lemma 3.3, one obtains

$$u_k\to u \text{ in } W^{1,p(.)}(\Omega) \text{ and } \nabla u_k\to\nabla u \text{ a.e. in } \Omega \text{ as } k\to\infty;$$

then,

$$a(x,u_k,\nabla u_k) \rightharpoonup a(x,u,\nabla u) \text{ in } (L^{p'(.)}(\Omega))^N \text{ as } k\to\infty \tag{4.28}$$

and

$$g_\epsilon(x,u_k,\nabla u_k) \rightharpoonup g_\epsilon(x,u,\nabla u) \text{ in } L^{p'(.)}(\Omega) \text{ as } k\to\infty. \tag{4.29}$$

Therefore, we can write $\chi_\epsilon = A_\epsilon u$, which end the proof of Lemma 4.4.
Since A_ϵ is bounded, coercive and pseudo-monotone, A_ϵ is surjective (see [27], Theorem 2.7). Therefore, for any $f\in (W^{1,p(.)}(\Omega))^*$, there exists at least one solution $u_\epsilon\in W^{1,p(.)}(\Omega)$ of the problem $(\mathcal{P}^{g_\epsilon}_{f,\beta_\epsilon})$, which complete the proof Theorem 4.4 . □

□

Step 2. The a priori estimate

Lemma 4.5. *Let $f\in L^\infty(\Omega)$ and $0<\epsilon\le 1$. If $u_\epsilon\in W^{1,p(.)}(\Omega)$ is a weak solution of $(\mathcal{P}^{g_\epsilon}_{f,\beta_\epsilon})$, then,*

$$\|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))\|_\infty \le \|f\|_\infty. \tag{4.30}$$

For any $k\ge 1$, there exists a constant $C_2>0$ not depending on k, such that

$$\|\nabla T_k(u_\epsilon)\|_{p(.)}\le C_2. \tag{4.31}$$

Proof. By using the test function $\varphi_{\delta,\epsilon} = \dfrac{1}{\delta}\Big[T_{k+\delta}(\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))) - T_k(\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)))\Big]$ in (4.13) where $\delta>0$, one obtains

$$\int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))\varphi_{\delta,\epsilon}dx + \int_\Omega a(x,u_\epsilon,\nabla u_\epsilon)\nabla\varphi_{\delta,\epsilon}dx + \int_\Omega g_\epsilon(x,u_\epsilon,\nabla u_\epsilon)\varphi_{\delta,\epsilon}dx$$

$$+\epsilon\int_\Omega |u_\epsilon|^{p(x)-2}u_\epsilon\varphi_{\delta,\epsilon}dx = \int_\Omega f\varphi_{\delta,\epsilon}dx. \tag{4.32}$$

Since $\nabla\varphi_{\delta,\epsilon} = \begin{cases} \frac{1}{\delta}(\beta'_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)))\nabla u_\epsilon & \text{if } k\le|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))|\le k+\delta,\\ 0 & \text{elsewhere}\end{cases}$
and β_ϵ is nondecreasing, by using (3.5), one gets

$$\int_\Omega a(x,u_\epsilon,\nabla u_\epsilon)\nabla\varphi_{\delta,\epsilon}dx = \frac{1}{\delta}\int_\Omega a(x,u_\epsilon,\nabla u_\epsilon)\beta'_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))\nabla u_\epsilon dx \ge 0.$$

Observing that $\varphi_{\delta,\epsilon}$ has the same sign as u_ϵ, one deduces from (3.9) that

$$\int_\Omega g_\epsilon(x,u_\epsilon,\nabla u_\epsilon)\varphi_{\delta,\epsilon}dx \geq 0.$$

Then,

$$\int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))\varphi_{\delta,\epsilon}dx \leq \int_\Omega f\varphi_{\delta,\epsilon}dx.$$

Therefore,

$$\int_{\{k+\delta\leq|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))|\}} \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))\varphi_{\delta,\epsilon}dx \leq \int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))\varphi_{\delta,\epsilon}dx \leq \int_\Omega f\varphi_{\delta,\epsilon}dx.$$

Since $\varphi_{\delta,\epsilon}=0$ on the set $\{|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))|<k\}$ and $|\varphi_{\delta,\epsilon}|\leq 1$, one has

$$\frac{1}{\delta}\int_{\{k+\delta\leq|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))|\}} \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))\varphi_{\delta,\epsilon}dx \leq \int_{\{k\leq|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))|\}}|f|dx. \tag{4.33}$$

Since $k\leq|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))|$ on the set $\{k+\delta\leq|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))|\}$, one has

$$\begin{aligned}
k\text{meas}\{k+\delta\leq|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))|\} &\leq \int_{\{k+\delta\leq|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))|\}}|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))|dx\\
&\leq \frac{1}{\delta}\int_{\{k\leq|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))|\}}\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))\\
&\times\left[T_{k+\delta}(\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)))-T_k(\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)))\right]dx\\
&\leq \int_{\{k\leq|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))|\}}|f|dx\\
&\leq \|f\|_{L^\infty(\Omega)}\text{meas}\{k\leq|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))|\}.
\end{aligned}$$

Letting $\delta\to 0$ and choosing $k>\|f\|_{L^\infty(\Omega)}$, one obtains

$$k\text{meas}\{k\leq|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))|\}\leq \|f\|_{L^\infty(\Omega)}\text{meas}\{k\leq|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))|\}.$$

It follows that $\text{meas}\{k\leq|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))|\}=0$ for any $k>\|f\|_{L^\infty(\Omega)}$. Thus,

$$\|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))\|_{L^\infty(\Omega)}\leq\|f\|_{L^\infty(\Omega)}.$$

Taking $T_k(u_\epsilon)$ as a test function in (4.13), one obtains

$$\int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))T_k(u_\epsilon)dx+\int_\Omega g_\epsilon(x,u_\epsilon,\nabla u_\epsilon)T_k(u_\epsilon)dx+\epsilon\int_\Omega|u_\epsilon|^{p(x)-2}u_\epsilon T_k(u_\epsilon)dx$$
$$+\int_\Omega a(x,u_\epsilon,\nabla u_\epsilon)\nabla T_k(u_\epsilon)dx=\int_\Omega fT_k(u_\epsilon)dx.$$

As the first tree terms of the left-hand side of the above equality are positives, one has

$$\int_\Omega a(x,u_\epsilon,\nabla u_\epsilon)\nabla T_k(u_\epsilon)dx=\int_{\{|u_\epsilon|\leq k\}}a(x,u_\epsilon,\nabla u_\epsilon)\nabla T_k(u_\epsilon)dx\leq\int_\Omega fT_k(u_\epsilon)dx.$$

From (3.5), one deduces that

$$\int_\Omega|\nabla T_k(u_\epsilon)|^{p(x)}dx\leq\frac{k\|f\|_\infty}{\lambda}.$$

Using Proposition 2.1, it follows that

$$\|\nabla T_k(u_\epsilon)\|^{\gamma}_{p(.)} \leq \frac{k\|f\|_\infty}{\lambda},$$

where

$$\gamma = \begin{cases} p_+ \text{ if } \|\nabla T_k(u_\epsilon)\|_{p(.)} \leq 1 \\ p_- \text{ if } \|\nabla T_k(u_\epsilon)\|_{p(.)} > 1. \end{cases}$$

Therefore,

$$\|\nabla T_k(u_\epsilon)\|_{p(.)} \leq C_2 \text{ for all } k \geq 1,$$

where $C_2 := \left(\frac{k\|f\|_\infty}{\lambda}\right)^{\frac{1}{\gamma}}$.

□

Lemma 4.6. *If u_ϵ is a solution of $(\mathcal{P}^{g_\epsilon}_{f,\beta_\epsilon})$, one has*

$$\int_\Omega (|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))| - k)^+ dx \leq \int_\Omega (|f| - k)^+ dx. \tag{4.34}$$

Proof. Applying the test function $\varphi = H^+_\delta(\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)) - k)$ in (4.13), one obtains

$$\int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))H^+_\delta(\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)) - k)dx + \int_\Omega a(x,u_\epsilon,\nabla u_\epsilon)\nabla H^+_\delta(\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)) - k)dx$$
$$+ \int_\Omega g_\epsilon(x,u_\epsilon,\nabla u_\epsilon)H^+_\delta(\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)) - k)dx + \epsilon\int_\Omega |u_\epsilon|^{p(x)-2}u_\epsilon H^+_\delta(\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)) - k)dx$$
$$= \int_\Omega f H^+_\delta(\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)) - k)dx.$$

Since β_ϵ is nondecreasing, using (3.5), one gets

$$\int_\Omega a(x,u_\epsilon,\nabla u_\epsilon)(H^+_\delta)'(\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)) - k)\beta'_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))\nabla u_\epsilon dx \geq 0.$$

Using the sign condition on g_ϵ, one has

$$\int_\Omega g_\epsilon(x,u_\epsilon,\nabla u_\epsilon)H^+_\delta(\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)) - k)dx \geq 0.$$

One also has

$$\epsilon\int_\Omega |u_\epsilon|^{p(x)-2}u_\epsilon H^+_\delta(\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)) - k)dx \geq 0.$$

Therefore,

$$\int_\Omega (\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)) - k)H^+_\delta(\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)) - k)dx \leq \int_\Omega (f-k)H^+_\delta(\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)) - k)dx.$$

Letting $\delta \to 0$ in the inequality above, one obtain

$$\int_\Omega (\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)) - k)^+ dx \leq \int_\Omega (f-k)^+ dx. \tag{4.35}$$

Reasoning similarly, one obtains

$$\int_\Omega (\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon) + k)^- dx \leq \int_\Omega (f+k)^- dx. \tag{4.36}$$

By combining (4.35) and (4.36), it follows (4.34). □

Proposition 4.5. [28] *Let u_ϵ be a weak solution of $(\mathcal{P}^{g_\epsilon}_{f,\beta_\epsilon})$. For any $k > 0$ large enough, one has*

$$\text{meas}\{|u_\epsilon| > k\} \leq \frac{\|f\|_1}{\min\{\beta_\epsilon(k), |\beta_\epsilon(-k)|\}} \tag{4.37}$$

and

$$meas\left\{|\nabla u_\epsilon| > k\right\} \leq \frac{C(k+1)}{k^{p^-}} + \frac{\|f\|_1}{\min\{\beta_\epsilon(k), |\beta_\epsilon(-k)|\}}, \tag{4.38}$$

where C is a positive constant.

Step 3. Convergence results

Proposition 4.6. *For any $k > 0$ and $f \in L^\infty(\Omega)$, if $u_\epsilon \in W^{1,p(.)}(\Omega)$ is a solution of problem $(\mathcal{P}^{g_\epsilon}_{f,\beta_\epsilon})$, then, there exists $b \in L^\infty(\Omega)$ such that*

$$\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)) \rightharpoonup b \text{ weakly-* in } L^\infty(\Omega) \text{ as } \epsilon \to 0. \tag{4.39}$$

Proof. Thanks to (4.30), there exists $b \in L^\infty(\Omega)$ such that (4.39) holds. □

Proposition 4.7.

(i) *For any $k > 0$, $T_k(u_\epsilon) \longrightarrow T_k(u)$ in $L^{p^-}(\Omega)$ and a.e. in Ω, as $\epsilon \to 0$.*

(ii) *There exists $u \in \mathcal{T}^{1,p(.)}(\Omega)$ such that $u \in dom(\beta)$ a.e. in Ω and*

$$u_\epsilon \to u \text{ in measure and a.e. in } \Omega, \text{ as } \epsilon \to 0.$$

Proof. It follows from (4.31) that the sequence $(\nabla T_k(u_\epsilon))_{\epsilon>0}$ is bounded in $L^{p(.)}(\Omega)$. Therefore, the sequence $(T_k(u_\epsilon))_{\epsilon>0}$ is bounded in $W^{1,p(.)}(\Omega)$. Then, there exists a subsequence still denoted $(T_k(u_\epsilon))_{\epsilon>0}$ and a measurable function $\sigma_k \in W^{1,p(.)}(\Omega)$ such that

$$\begin{cases} T_k(u_\epsilon) \rightharpoonup \sigma_k \text{ in } W^{1,p(.)}(\Omega) \text{ as } \epsilon \to 0, \\ T_k(u_\epsilon) \to \sigma_k \text{ in } L^{p^-}(\Omega) \text{ and a.e. in } \Omega \text{ as } \epsilon \to 0. \end{cases} \tag{4.40}$$

The sequence $(u_\epsilon)_{\epsilon>0}$ is a Cauchy sequence in measure in Ω.
Indeed, let $s > 0$ and set $E_{\epsilon_1} = \{|u_{\epsilon_1}| > k\}$, $E_{\epsilon_2} = \{|u_{\epsilon_2}| > k\}$ and $E_{\epsilon_1,\epsilon_2} = \{|T_k(u_{\epsilon_1}) + T_k(u_{\epsilon_2})| > k\}$, where $k > 0$ is a real number to be chosen later.
One has

$$\{|u_{\epsilon_1} - u_{\epsilon_2}| > s\} \subset E_{\epsilon_1} \cup E_{\epsilon_2} \cup E_{\epsilon_1,\epsilon_2},$$

which implies that

$$\text{meas}\{|u_{\epsilon_1} - u_{\epsilon_2}| > s\} \leq \text{meas}(E_{\epsilon_1}) + \text{meas}(E_{\epsilon_2}) + \text{meas}(E_{\epsilon_1,\epsilon_2}). \tag{4.41}$$

Let $\theta > 0$. By using (4.37), there exists $k_0 = k_0(\theta)$ such that

$$\forall k \geq k_0(\theta),\ \text{meas}\{|u_{\epsilon_1}| > k\} \leq \frac{\theta}{3} \text{ and } \text{meas}\{|u_{\epsilon_2}| > k\} \leq \frac{\theta}{3}. \tag{4.42}$$

Since the sequence $(T_k(u_\epsilon))_{\epsilon>0}$ converges strongly in $L^{p^-}(\Omega)$, then it is a Cauchy sequence in $L^{p^-}(\Omega)$. Consequently, for any $s > 0$ and $\theta > 0$, there exists $n_0 = n_0(\theta, s)$ such that : $\forall \epsilon_1, \epsilon_2 \geq n_0$, one has

$$\left(\int_\Omega |T_k(u_{\epsilon_1}) - T_k(u_{\epsilon_2})|^{p^-} dx\right)^{\frac{1}{p^-}} \leq \left(\frac{\theta s^{p^-}}{3}\right)^{\frac{1}{p^-}}.$$

Then, one deduces that

$$\text{meas}(E_{\epsilon_1,\epsilon_2}) \leq \frac{1}{p^-}\int_\Omega |T_k(u_{\epsilon_1}) - T_k(u_{\epsilon_2})|^{p^-} dx \leq \frac{\theta}{3}. \tag{4.43}$$

Now, we fix $k = k_0(\theta)$ and $n_0 = n_0(\theta, s)$. One can deduces from (4.41) and (4.43) that

$$\text{meas}\{|u_{\epsilon_1} - u_{\epsilon_2}| > \delta\} \leq \theta \text{ for any } \epsilon_1, \epsilon_2 \geq n_0,$$

which means that $(u_\epsilon)_{\epsilon>0}$ is a Cauchy sequence in measure and there exists a subsequence still denoted $(u_\epsilon)_{\epsilon>0}$ and some measurable function u such that

$$u_\epsilon \longrightarrow u \text{ a.e. in } \Omega.$$

Hence, $\sigma_k = T_k(u)$ a.e. in Ω and so $u \in \mathcal{T}^{1,p(.)}(\Omega)$. Therefore, $T_k(u) \in \text{dom}\beta$ a.e. in Ω for any $k > 0$. Consequently, $u \in \text{dom}\beta$ a.e. in Ω (see [3]). □

We need the following strong convergence results.

Proposition 4.8. *If u_ϵ is a solution of the approximate problem $(\mathcal{P}^{g_\epsilon}_{f,\beta_\epsilon})$, there exists a measurable function u and a subsequence of $(u_\epsilon)_{\epsilon>0}$ such that*

$$T_k(u_\epsilon) \to T_k(u) \text{ strongly in } W^{1,p(.)}(\Omega), \tag{4.44}$$

$$\nabla u_\epsilon \to \nabla u \text{ a.e. in } \Omega, \tag{4.45}$$

$$a(x, u_\epsilon, \nabla u_\epsilon) \rightharpoonup a(x, u, \nabla u) \text{ weakly in } (L^{p'(.)}(\Omega))^N \tag{4.46}$$

and

$$g_\epsilon(x, u_\epsilon, \nabla u_\epsilon) \to g(x, u, \nabla u) \text{ strongly in } L^1(\Omega). \tag{4.47}$$

Proof. For any $k \geq 0$, we shall use as test function

$$\varphi_\epsilon = \varphi(T_k(u_\epsilon) - T_k(u)),$$

where

$$\varphi(s) = se^{\alpha s^2} \text{ and } \alpha = \left(\frac{b(k)}{\lambda}\right)^2.$$

It is well known (see [10], Lemma 1) that

$$\varphi'(s) - \frac{b(k)}{\lambda}|\varphi(s)| \geq \frac{1}{2}, \quad \forall s \in \mathbb{R}. \tag{4.48}$$

Taking in (4.13) the test function φ_ϵ, one obtains

$$\int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))\varphi_\epsilon dx + \int_\Omega a(x, u_\epsilon, \nabla u_\epsilon)\nabla\varphi_\epsilon dx + \int_\Omega g_\epsilon(x, u_\epsilon, \nabla u_\epsilon)\varphi_\epsilon dx$$
$$+\epsilon \int_\Omega |u_\epsilon|^{p(x)-1}|\varphi_\epsilon| dx = \int_\Omega f\varphi_\epsilon dx.$$

Since $g_\epsilon(x, u_\epsilon, \nabla u_\epsilon)\varphi_\epsilon \geq 0$ and $\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))\varphi_\epsilon \geq 0$ on $\{|u_\epsilon| > k\}$, one deduces that

$$\int_{\{|u_\epsilon|\leq k\}} \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))\varphi_\epsilon dx + \int_\Omega a(x, u_\epsilon, \nabla u_\epsilon)\nabla\varphi_\epsilon dx + \int_{\{|u_\epsilon|\leq k\}} g_\epsilon(x, u_\epsilon, \nabla u_\epsilon)\varphi_\epsilon dx$$
$$+\epsilon \int_\Omega |u_\epsilon|^{p(x)-2}u_\epsilon\varphi_\epsilon dx \leq \int_\Omega f_\epsilon\varphi_\epsilon dx. \tag{4.49}$$

Since $|u_\epsilon|^{p(x)-1} \longrightarrow |u|^{p(x)-1}$ in $L^{p'(.)}(\Omega)$ and $\varphi_\epsilon \longrightarrow 0$ in $L^\infty(\Omega)$ as $\epsilon \to 0$, one has

$$\left|\epsilon \int_\Omega |u_\epsilon|^{p(x)-2}u_\epsilon\varphi_\epsilon dx\right| \leq \epsilon \int_\Omega |u_\epsilon|^{p(x)-1}|\varphi_\epsilon| dx$$
$$\longrightarrow 0 \text{ as } \epsilon \to 0.$$

Then, one can write

$$\epsilon \int_\Omega |u_\epsilon|^{p(x)-2} u_\epsilon \varphi_\epsilon dx = \eta^1(\epsilon), \tag{4.50}$$

where $\eta^1(\epsilon)$ is a sequence of real numbers which converges to zero as ϵ goes to 0. In the sequel we will denote by $\eta^i(\epsilon), i = 1, 2, ...$ such sequences.
According to Lebesgue generalized convergence Theorem, one has

$$\int_\Omega f \varphi_\epsilon dx = \eta^1(\epsilon). \tag{4.51}$$

Since the sequence $(\chi_{\{|u_\epsilon| \le k\}} \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)))_{\epsilon > 0}$ is uniformly bounded, one deduces from the Lebesgue dominated convergence theorem that

$$\int_{\{|u_\epsilon| \le k\}} \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)) \varphi_\epsilon dx = \eta^3(\epsilon). \tag{4.52}$$

We now define, for any s and k in $\mathbb{R}$, with $k \ge 0$, $G_k(s) = s - T_k(s)$. Then, using the assumption (3.8) and the definition of function $G_k(.)$, one can decompose the second term of (4.49) as follows.

$$\begin{aligned} &\int_\Omega a(x, u_\epsilon, \nabla u_\epsilon) \cdot \nabla(T_k(u_\epsilon) - T_k(u)) \varphi'(T_k(u_\epsilon) - T_k(u)) dx \\ &= \int_\Omega a(x, T_k(u_\epsilon), \nabla T_k(u_\epsilon)) \cdot \nabla(T_k(u_\epsilon) - T_k(u)) \varphi'(T_k(u_\epsilon) - T_k(u)) dx \\ &+ \int_\Omega a(x, u_\epsilon, \nabla G_k(u_\epsilon)) \cdot \nabla(T_k(u_\epsilon) - T_k(u)) \varphi'(T_k(u_\epsilon) - T_k(u)) dx. \end{aligned} \tag{4.53}$$

Since $\nabla T_k(u_\epsilon)$ is zero where $\nabla G_k(u_\epsilon)$ is different to zero, and conversely, one has

$$\begin{aligned} &\int_\Omega a(x, u_\epsilon, \nabla G_k(u_\epsilon)) \cdot \nabla(T_k(u_\epsilon) - T_k(u)) \varphi'(T_k(u_\epsilon) - T_k(u)) dx \\ &= -\int_\Omega a(x, u_\epsilon, \nabla G_k(u_\epsilon)) \cdot \nabla T_k(u) \varphi'(T_k(u_\epsilon) - T_k(u)) dx. \end{aligned}$$

Since $\nabla T_k(u_\epsilon) \equiv 0$ on the set $\{|u_\epsilon| \ge k\}$, one has

$$\nabla T_k(u) \chi_{\{|u_\epsilon| \ge k\}} \to 0, \quad \text{a.e. in } \Omega \text{ as } \epsilon \to 0.$$

Using the fact that $\nabla T_k(u_\epsilon) \in (L^{p'(\cdot)}(\Omega))^N$, one deduces from the Lebesgue dominated convergence theorem that

$$\nabla T_k(u) \chi_{\{|u_\epsilon| \ge k\}} \to 0 \text{ strongly in } (L^{p'(\cdot)}(\Omega))^N, \text{ as } \epsilon \to 0.$$

Having in mind that $(a(x, u_\epsilon, \nabla G_k(u_\epsilon)))_{\epsilon > 0}$ is bounded in $(L^{p'(\cdot)}(\Omega))^N$, one obtains

$$\int_\Omega a(x, u_\epsilon, \nabla G_k(u_\epsilon)) \cdot \nabla(T_k(u_\epsilon) - T_k(u)) \varphi'(T_k(u_\epsilon) - T_k(u)) dx = \eta^3(\epsilon) \to 0. \tag{4.54}$$

Now we decompose the second term of (4.53) as follows.

$$\begin{aligned} &\int_\Omega a(x, u_\epsilon, \nabla u_\epsilon) \cdot \nabla(T_k(u_\epsilon) - T_k(u)) \varphi'(T_k(u_\epsilon) - T_k(u)) dx \\ &= \int_\Omega [a(x, T_k(u_\epsilon), \nabla T_k(u_\epsilon)) - a(x, T_k(u_\epsilon), \nabla T_k(u))] \cdot \nabla(T_k(u_\epsilon) - T_k(u)) \varphi'(T_k(u_\epsilon) - T_k(u)) dx \\ &+ \int_\Omega a(x, T_k(u_\epsilon), \nabla T_k(u)) \cdot \nabla(T_k(u_\epsilon) - T_k(u)) \varphi'(T_k(u_\epsilon) - T_k(u)) dx. \end{aligned}$$

Since $T_k(u_\epsilon)$ converges weakly to $T_k(u)$ in $W_0^{1,p'(.)}(\Omega)$ as $\epsilon \to 0$, $\lim\limits_{\epsilon\to 0} \varphi'(T_k(u_\epsilon) - T_k(u)) = 0$ and as the sequence $(a(x, T_k(u_\epsilon), \nabla T_k(u)))_{\epsilon>0}$ is bounded in $(L^{p'(.)}(\Omega))^N$, one deduces that

$$\int_\Omega a(x, T_k(u_\epsilon), \nabla T_k(u)) \cdot \nabla(T_k(u_\epsilon) - T_k(u))\varphi'(T_k(u_\epsilon) - T_k(u))dx = \eta^4(\epsilon). \tag{4.55}$$

Thus, putting together (4.54) and (4.55), follows

$$\int_\Omega a(x, u_\epsilon, \nabla u_\epsilon) \cdot \nabla(T_k(u_\epsilon) - T_k(u))\varphi'(T_k(u_\epsilon) - T_k(u))dx = \tag{4.56}$$

$$\int_\Omega [a(x, T_k(u_\epsilon), \nabla T_k(u_\epsilon)) - a(x, T_k(u_\epsilon), \nabla T_k(u))] \cdot \nabla(T_k(u_\epsilon) - T_k(u))\varphi'(T_k(u_\epsilon) - T_k(u))dx$$

$$+\eta^5(\epsilon).$$

On the other hand, one has

$$\begin{aligned}\left|\int_{\{|u_\epsilon|\leq k\}} g_\epsilon(x, u_\epsilon, \nabla u_\epsilon)\varphi_\epsilon dx\right| &\leq \int_{\{|u_\epsilon|\leq k\}} |g_\epsilon(x, u_\epsilon, \nabla u_\epsilon)||\varphi_\epsilon|dx \\ &\leq b(k)\int_\Omega (c(x) + |\nabla T_k(u_\epsilon)|^{p(x)})|\varphi_\epsilon|dx.\end{aligned}$$

Since c belongs to $L^1(\Omega)$ and $\lim\limits_{\epsilon\to 0} \varphi_\epsilon = \varphi(0) = 0$, one has

$$\int_\Omega c(x)|\varphi_\epsilon|dx = \eta^5(\epsilon). \tag{4.57}$$

Then, using (3.5), one has

$$\left|\int_{\{|u_\epsilon|\leq k\}} g_\epsilon(x, u_\epsilon, \nabla u_\epsilon)\varphi_\epsilon dx\right| \leq \frac{b(k)}{\lambda}\int_\Omega a(x, T_k(u_\epsilon), \nabla T_k(u_\epsilon)) \cdot \nabla T_k(u_\epsilon)|\varphi_\epsilon|dx + \eta^5(\epsilon). \tag{4.58}$$

Now we adding and subtracting to the above inequality the term

$$\frac{b(k)}{\lambda}\int_\Omega a(x, T_k(u_\epsilon), \nabla T_k(u_\epsilon)) \cdot \nabla(T_k(u_\epsilon) - T_k(u))|\varphi_\epsilon|dx,$$

to obtain

$$\begin{cases} \left|\int_{\{|u_\epsilon|\leq k\}} g_\epsilon(x, u_\epsilon, \nabla u_\epsilon)\varphi_\epsilon dx\right| \\ \leq \frac{b(k)}{\lambda}\int_\Omega \Big(a(x, T_k(u_\epsilon), \nabla T_k(u_\epsilon) - a(x, T_k(u_\epsilon), \nabla T_k(u)\Big) \cdot \nabla(T_k(u_\epsilon) - T_k(u))|\varphi_\epsilon|dx \\ +\frac{b(k)}{\lambda}\int_\Omega a(x, T_k(u_\epsilon), \nabla T_k(u)) \cdot \nabla T_k(u)|\varphi_\epsilon|dx + \eta^5(\epsilon). \end{cases} \tag{4.59}$$

Since $(a(x, T_k(u_\epsilon), \nabla T_k(u)))_{\epsilon>0}$ is bounded in $(L^{p'(.)}(\Omega))^N$ and φ_ϵ converges to zero as $\epsilon \to 0$, one has

$$\frac{b(k)}{\lambda}\int_\Omega a(x, T_k(u_\epsilon), \nabla T_k(u)) \cdot \nabla T_k(u)|\varphi_\epsilon|dx = \eta^6(\epsilon).$$

Then, one deduces that

$$(4.60)\qquad \begin{cases} \left| \int_{\{|u_\epsilon|\leq k\}} g_\epsilon(x,u_\epsilon,\nabla u_\epsilon)\varphi_\epsilon dx \right| \\ \leq \frac{b(k)}{\lambda} \int_\Omega \Big(a(x,T_k(u_\epsilon),\nabla T_k(u_\epsilon)) - a(x,T_k(u),\nabla T_k(u)) \Big) \\ \times \nabla(T_k(u_\epsilon)-T_k(u))|\varphi_\epsilon| dx + \eta^7(\epsilon). \end{cases}$$

Putting (4.49) and (4.60) together, one deduces that

$$\int_\Omega [a(x,T_k(u_\epsilon),\nabla T_k(u_\epsilon)) - a(x,T_k(u_\epsilon),\nabla T_k(u))]\cdot\nabla(T_k(u_\epsilon)-T_k(u))$$

$$(4.61)\qquad \times[\varphi'_\epsilon - \frac{b(k)}{\lambda}|\varphi_\epsilon|]dx \leq \eta^7(\epsilon).$$

Then, using (4.48), one obtains

$$0 \leq \frac{1}{2}\int_\Omega [a(x,T_k(u_\epsilon),\nabla T_k(u_\epsilon)) - a(x,T_k(u_\epsilon),\nabla T_k(u))]\cdot\nabla(T_k(u_\epsilon)-T_k(u))dx \leq \eta^7(\epsilon).$$

Therefore,

$$(4.62)\qquad \lim_{\epsilon\to 0}\int_\Omega [a(x,T_k(u_\epsilon),\nabla T_k(u_\epsilon)) - a(x,T_k(u_\epsilon),\nabla T_k(u))]\cdot\nabla(T_k(u_\epsilon)-T_k(u))dx = 0.$$

Using Lemma 3.3, as $\epsilon\to 0$, one has

$$(4.63)\qquad T_k(u_\epsilon)\to T_k(u) \text{ in } W^{1,p(.)}(\Omega).$$

The strong convergence of $T_k(u_\epsilon)$ implies that for some subsequence, still denoted by u_ϵ,

$$\nabla u_\epsilon \to \nabla u \text{ a.e. in } \Omega.$$

Since the functions $a(x,.,.)$ and $g(x,.,.)$ are continuous for $a.e.\ x$ in Ω, one has

$$(4.64)\qquad a(x,u_\epsilon,\nabla u_\epsilon)\to a(x,u,\nabla u) \text{ a.e. in } \Omega$$

and

$$(4.65)\qquad g_\epsilon(x,u_\epsilon,\nabla u_\epsilon)\to g(x,u,\nabla u) \text{ a.e. in } \Omega.$$

Since $(a(x,u_\epsilon,\nabla u_\epsilon))_{\epsilon>0}$ is bounded in $(L^{p'(.)}(\Omega))^N$, using (4.64) and Lemma 2.1, one gets

$$a(x,u_\epsilon,\nabla u_\epsilon)\rightharpoonup a(x,u,\nabla u) \text{ weakly in } (L^{p'(.)}(\Omega))^N \text{ as } \epsilon\to 0.$$

□

Remark 4.2. *One can also deduce from the above step that*

$$\epsilon|u_\epsilon|^{p(x)-2}u_\epsilon \longrightarrow 0 \ \textit{ a.e. in } \Omega \textit{ as } \epsilon\to 0.$$

Note that the above strong convergence is not sufficient to pass to the limit, so we need the following results.

Proposition 4.9.

$$(4.66)\qquad g_\epsilon(x,u_\epsilon,\nabla u_\epsilon)\to g(x,u,\nabla u) \textit{ and } \epsilon|u_\epsilon|^{p(x)-2}u_\epsilon \longrightarrow 0 \textit{ strongly in } L^1(\Omega) \textit{ as } \epsilon\to 0.$$

Proof. One has

$$g_\epsilon(x,u_\epsilon,\nabla u_\epsilon)\to g(x,u,\nabla u) \text{ and } \epsilon|u_\epsilon|^{p(x)-2}u_\epsilon \longrightarrow 0 \text{ a.e. in } \Omega.$$

According to Vitali's theorem, it suffices to show that the sequences $(g_\epsilon(x,u_\epsilon,\nabla u_\epsilon))_{\epsilon>0}$ and $(\epsilon|u_\epsilon|^{p(x)-2}u_\epsilon)_{\epsilon>0}$ are uniformly equi-integrable.
Taking $\varphi=T_1(u_\epsilon-T_n(u_\epsilon))$ as test function in (4.13), one gets

$$\int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))T_1(u_\epsilon-T_n(u_\epsilon))dx+\int_\Omega a(x,u_\epsilon,\nabla u_\epsilon).\nabla[T_1(u_\epsilon-T_n(u_\epsilon))]dx \tag{4.67}$$
$$+\int_\Omega g_\epsilon(x,u_\epsilon,\nabla u_\epsilon)T_1(u_\epsilon-T_n(u_\epsilon))dx+\epsilon\int_\Omega |u_\epsilon|^{p(x)-2}u_\epsilon T_1(u_\epsilon-T_n(u_\epsilon))dx$$
$$=\int_\Omega fT_1(u_\epsilon-T_n(u_\epsilon))dx.$$

Since $T_1(u_\epsilon-T_n(u_\epsilon))$ has the same sign with u_ϵ and $\nabla T_1(u_\epsilon-T_n(u_\epsilon))=\nabla u_\epsilon\chi_{[n<u_\epsilon\leq n+1]}$, the first and the second terms of (4.67) are nonnegative. Then, one deduces that

$$\int_{\{|u_\epsilon|>n\}} g_\epsilon(x,u_\epsilon,\nabla u_\epsilon)T_1(u_\epsilon-T_n(u_\epsilon))dx+\epsilon\int_{\{|u_\epsilon|>n\}}|u_\epsilon|^{p(x)-2}u_\epsilon T_1(u_\epsilon-T_n(u_\epsilon))dx$$
$$\leq \int_{\{|u_\epsilon|>n\}}|f|dx.$$

Since $\{|u_\epsilon|\geq n+1\}\subset\{|u_\epsilon|>n\}$, one deduces from the above inequality that

$$\int_{\{|u_\epsilon|\geq n+1\}}|g_\epsilon(x,u_\epsilon,\nabla u_\epsilon)|dx+\epsilon\int_{\{|u_\epsilon|\geq n+1\}}|u_\epsilon|^{p(x)-1}dx\leq \|f\|_\infty \text{meas}(\{|u_\epsilon|>n\}).$$

Since $\text{meas}(\{|u_\epsilon|>n\})\longrightarrow 0$ as $n\to\infty$, one deduces that

$$\lim_{n\to\infty}\limsup_{\epsilon\to 0}\left(\int_{\{|u_\epsilon|\geq n+1\}}|g_\epsilon(x,u_\epsilon,\nabla u_\epsilon)|dx+\epsilon\int_{\{|u_\epsilon|\geq n+1\}}|u_\epsilon|^{p(x)-1}dx\right)=0,$$

thus, for any $\gamma>0$, there exists $h(\gamma)>0$ such that

$$\int_{\{|u_\epsilon|\geq h(\gamma)\}}|g_\epsilon(x,u_\epsilon,\nabla u_\epsilon)|dx+\epsilon\int_{\{|u_\epsilon|\geq h(\gamma)\}}|u_\epsilon|^{p(x)-1}dx\leq\frac{\gamma}{2}. \tag{4.68}$$

For any measurable subset $A\subset\Omega$, one has

$$\int_A|g_\epsilon(x,u_\epsilon,\nabla u_\epsilon)|dx+\epsilon\int_A|u_\epsilon|^{p(x)-1}dx\leq b(h(\gamma))\int_A(C(x)+|\nabla T_{h(\gamma)}(u_\epsilon)|^{p(x)})dx$$
$$+\int_{\{|u_\epsilon|\geq h(\gamma)\}}|g_\epsilon(x,u_\epsilon,\nabla u_\epsilon)|dx$$
$$+\int_A|T_{h(\gamma)}(u_\epsilon)|^{p(x)-1}dx$$
$$+\epsilon\int_{\{|u_\epsilon|\geq h(\gamma)\}}|u_\epsilon|^{p(x)-1}dx. \tag{4.69}$$

According to (4.44), there exists $\theta(\gamma)>0$ such that, for all $A\subseteq\Omega$ with $\text{meas}(A)\leq\theta(\gamma)$,

$$b(h(\gamma))\int_A(C(x)+|\nabla T_{h(\gamma)}(u_\epsilon)|^{p(x)})dx+\epsilon\int_A|u_\epsilon|^{p(x)-1}dx\leq\frac{\gamma}{2}. \tag{4.70}$$

Combining (4.68), (4.69) and (4.70), one has

$$\int_A|g_\epsilon(x,u_\epsilon,\nabla u_\epsilon)|dx+\epsilon\int_A|u_\epsilon|^{p(x)-1}dx\leq\gamma, \tag{4.71}$$

for all $A \subseteq \Omega$ such that meas$(A) \leq \theta(\gamma)$.
Now, we conclude that the sequences $(g_\epsilon(x, u_\epsilon, \nabla u_\epsilon))_{\epsilon>0}$ and $(\epsilon|u_\epsilon|^{p(x)-2}u_\epsilon)_{\epsilon>0}$ are equi-integrable. Then, from Vitali's theorem one deduces (4.66).
Based on the above convergence results, we pass to the limit in (4.13) as $\epsilon \to 0$ to get

$$\int_\Omega b\varphi dx + \int_\Omega a(x, u, \nabla u)\nabla\varphi dx + \int_\Omega g(x, u, \nabla u)\varphi dx = \int_\Omega f\varphi dx.$$

We conclude the proof of Theorem 4.3 by writing $u \in D(\beta)$ and $b \in \beta(u)$ almost everywhere in Ω (see [3]). □

4.2. **Proof of the case** $f \in L^1(\Omega)$**.**
This section is devoted to the proof of Theorem 4.3, Theorem 5.5 and Theorem 5.6 for the case where the datum f belongs to $L^1(\Omega)$. To this end, we divide our arguments into several steps.

Step 1. The approximated problem For each $n \in \mathbb{N}$, we consider the following approximated problem.

$$(\mathcal{P}^g_{f_n,\beta}) \begin{cases} \beta(u_n) - diva(x, u_n, \nabla u_n) + g(x, u_n, \nabla u_n) \ni f_n \text{ in } \Omega, \\ \\ a(x, u_n, \nabla u_n) \cdot \nu = 0 \qquad \text{on } \partial\Omega, \end{cases}$$

where f_n is a sequence of L^∞-functions which converges strongly to f in $L^1(\Omega)$ and $|f_n| \leq |f|$. For example, one can choose $f_n = T_n(f)$.
Thanks to Section 4, there exists a solution $(u_n, b_n) \in W^{1,p(\cdot)}(\Omega) \times L^\infty(\Omega)$ of $(\mathcal{P}^g_{f_n,\beta})$ such that

$$\int_\Omega b_n\varphi dx + \int_\Omega a(x, u_n, \nabla u_n)D\varphi dx + \int_\Omega g(x, u_n, \nabla u_n)\varphi dx = \int_\Omega f_n\varphi dx, \tag{4.72}$$

for all $\varphi \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

Step 2. The a priori estimates

Lemma 4.7. *For $n \in \mathbb{N}$, let $(u_n, b_n) \in W^{1,p(\cdot)}(\Omega) \times L^\infty(\Omega)$ be a solution of $(\mathcal{P}^g_{f_n,\beta})$. For any $k \geq 1$, there exists a constant $C_2 > 0$ not depending on k such that*

$$\|\nabla T_k(u_n)\|_{p(.)} \leq C_2 k^{\frac{1}{\gamma}} \tag{4.73}$$

and

$$\|b_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}. \tag{4.74}$$

Proof. Taking $T_k(u_n)$ as a test function in (4.72), one obtains

$$\int_\Omega b_n T_k(u_n)dx + \int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n)dx + \int_\Omega g(x, u_n, \nabla u_n)T_k(u_n)dx$$

$$= \int_\Omega fT_k(u_n)dx. \tag{4.75}$$

Since the first term on the left hand side of (4.75) is nonnegative and g verifies the sign condition, from (3.5), one deduces that

$$\int_\Omega |\nabla Tk(u_n)|^{p(x)}dx \leq \frac{k\|f\|_{L^1(\Omega)}}{\lambda}.$$

Using Proposition 2.1, one gets

$$\|\nabla Tk(u_n)\|^\gamma_{p(.)} \leq \frac{k\|f\|_{L^1(\Omega)}}{\lambda}.$$

Then,

$$\|\nabla Tk(u_n)\|_{p(x)} \leq C_6 k^{\frac{1}{\gamma}} \text{ for all } k \geq 1,$$

where $C_6 := \left(\frac{\|f\|_{L^1(\Omega)}}{\lambda}\right)^{\frac{1}{\gamma}}$.

Since $\int_\Omega b_n T_k(u_n)dx \geq 0$ and $\int_\Omega g(x, u_n, \nabla u_n) T_k(u_n)dx \geq 0$, we deduce from (4.75) that

$$\int_\Omega b_n T_k(u_n)dx \leq \int_\Omega f T_k(u_n)dx \leq k\|f\|_{L^1(\Omega)}.$$

Dividing the above inequality by $k > 0$, one obtains

$$\int_\Omega b_n \frac{1}{k} T_k(u_n)dx \leq \|f\|_{L^1(\Omega)}.$$

Since $b_n \in \beta(u_n)$ a.e. in Ω and $\lim_{k\to\infty} \frac{1}{k} T_k(u_n) = \text{sign}_0(u_n)$, we pass to the limit as $k \to \infty$ to get

$$\int_\Omega |b_n|dx \leq \|f\|_{L^1(\Omega)}.$$

□

Step 3. Basic convergence results and Passage to the limit

Lemma 4.8. *For $n \in \mathbb{N}$, let $(u_n, b_n) \in W^{1,p(.)}(\Omega) \times L^\infty(\Omega)$ be a solution of $(\mathcal{P}^g_{f_n,\beta})$. Then, as $n \to \infty$, one has*

$$b_n \rightharpoonup b \text{ weakly in } L^1(\Omega), \tag{4.76}$$

$$u_n \to u \text{ a.e. in } \Omega, \tag{4.77}$$

$$T_k(u_n) \rightharpoonup T_k(u) \text{ in } W^{1,p(.)}(\Omega), \tag{4.78}$$

$$T_k(u_n) \to T_k(u) \text{ in } L^{p(.)}(\Omega) \text{ and a.e. in } \Omega. \tag{4.79}$$

Proof. Let $(u_n^\epsilon, b_n^\epsilon)$ be a solution of the following problem.

$$\begin{cases} \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_n^\epsilon)) - diva(x, u_n^\epsilon, \nabla u_n^\epsilon) + g(x, u_n^\epsilon, \nabla u_n^\epsilon) = f_n & \text{in } \Omega \\ a(x, u_n^\epsilon, \nabla u_n^\epsilon) \cdot \nu = 0 & \text{on } \partial\Omega. \end{cases}$$

The proof of (4.76) follows the same line as in [3].
From Lemma 4.6, one has

$$\int_\Omega (|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_n^\epsilon))| - k)^+ dx \leq \int_\Omega (|f_n| - k)^+ dx. \tag{4.80}$$

Since $\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_n^\epsilon)) \rightharpoonup b_n$ in $L^\infty(\Omega)$ as ϵ goes to 0, one gets

$$\int_\Omega (|b_n| - k)^+ dx \leq \int_\Omega (|f_n| - k)^+ dx. \tag{4.81}$$

Remark 4.3. *One has*

$$\lim_{k\to\infty} meas(\{|f_n| \geq k\}) = 0, \tag{4.82}$$

$$\lim_{k\to\infty} \int_\Omega (|f_n| - k)^+ dx = 0 \tag{4.83}$$

and

(4.84) $$\lim_{k\to\infty} k\,meas(\{|b_n|\geq k\})=0.$$

Indeed,

since $\int_\Omega |f_n|dx \leq \int_\Omega |f|dx = \|f\|_1$, *passing to the limit as* $k\to\infty$ *in the inequality*

$$meas(\{|f_n|\geq k\}) \leq \frac{1}{k}\int_\Omega |f_n|dx,$$

one obtains (4.82).

Since $A:=(f_n)_{n\in\mathbb{N}}\subset L^1(\Omega)$ *and* $|f_n|\leq |f|\in L^1(\Omega)$ *for any* $n\in\mathbb{N}$, *according to Proposition 2.4, the sequence* $(f_n)_{n\in\mathbb{N}}$ *is uniformly integrable.*

$$\begin{aligned}\int_\Omega (|f_n|-k)^+dx &= \int_{\{|f_n|\geq k\}} |f_n|dx - k\,meas(\{|f_n|\geq k\})\\ &\leq \int_{\{|f_n|\geq k\}} |f_n|dx\\ &\leq \sup_{f_n\in A}\int_{\{|f_n|\geq k\}} |f_n|dx.\end{aligned}$$

Then, passing to the limit in the above inequality as $k\to\infty$, *one deduces* (4.83).

Claim: The sequence $(b_n)_{n\in\mathbb{N}}$ converges weakly to a function b in $L^1(\Omega)$ as $n\to\infty$.
Indeed,

$$\begin{aligned}\int_{\{|f_n|\geq k\}} |b_n|dx &= \int_\Omega (|b_n|-k)^+dx + k\text{meas}(\{|b_n|\geq k\})\\ &\leq \int_\Omega (|f_n|-k)^+dx + k\text{meas}(\{|b_n|\geq k\}).\end{aligned}$$

Using (4.84), one has

$$\lim_{k\to\infty}\sup_{f_n\in\mathcal{F}}\left(\int_{\{|f_n|\geq k\}}|b_n|dx\right)=0.$$

Therefore, the sequence $(b_n)_{n\in\mathbb{N}}$ is uniformly integrable. Then, one can deduce from Theorem 2.2 that it is relatively weakly compact in $L^1(\Omega)$.

Therefore, one deduces that there exists a subsequence still denoted $(b_n)_{n\in\mathbb{N}}$ such that

$$b_n \rightharpoonup b \text{ weakly in } L^1(\Omega) \text{ as } n\to\infty.$$

Thanks to (4.73), the convergences (4.77), (4.78) and (4.79) hold (see the proofs of Proposition 4.7 and Proposition 4.8). □

Lemma 4.9. *For* $n\in\mathbb{N}$, *let* $(u_n,b_n)\in W^{1,p(.)}(\Omega)\times L^\infty(\Omega)$ *be a solution of* $(\mathcal{P}^g_{f_n,\beta})$. *As* $n\to\infty$, *one has*

(4.85) $$T_k(u_n)\to T_k(u) \textit{ strongly in } W_0^{1,p(.)}(\Omega),$$

(4.86) $$\nabla u_n \to \nabla u \textit{ a.e. in } \Omega,$$

(4.87) $$a(x,u_n,\nabla u_n)\to a(x,u,\nabla u) \textit{ a.e. in } \Omega,$$

(4.88) $$g_\epsilon(x,u_n,\nabla u_n)\to g(x,u,\nabla u) \textit{ a.e. in } \Omega$$

and

(4.89) $$a(x,u_n,\nabla u_n)\rightharpoonup a(x,u,\nabla u) \textit{ weakly in } (L^{p'(.)}(\Omega))^N.$$

Proof. For any $k \geq 0$, we define the function

$$\varphi_n = \varphi(T_k(u_n) - T_k(u)),$$

where

$$\varphi(s) = se^{\alpha s^2} \text{ and } \alpha = \left(\frac{b(k)}{\lambda}\right)^2.$$

By taking φ_n as test function in (4.72), we get

$$\int_\Omega b_n\varphi_n dx + \int_\Omega a(x, u_n, \nabla u_n)\nabla\varphi_n dx + \int_\Omega g(x, u_n, \nabla u_n)\varphi_n dx = \int_\Omega f_n\varphi_n dx. \tag{4.90}$$

Let us denote by $\eta^1(n),\ \eta^2(n), ...,$ various sequences of real numbers which converge to zero as $n \to \infty$.
Since $\varphi_n \overset{*}{\rightharpoonup} 0$ in $L^\infty(\Omega)$ and $f_n \to f$ in $L^1(\Omega)$ as $n \to \infty$, one has

$$\int_\Omega f_n\varphi_n dx = \eta^1(n).$$

As

$$\int_\Omega b_n\varphi_n dx = \int_{\{|u_n|\leq k\}} b_n\varphi_n dx + \int_{\{|u_n|>k\}} b_n\varphi_n dx \tag{4.91}$$

and $b_n \in \beta(u_n)$, the second term of (4.91) is nonnegative.
Notice that the function $\chi_{\{|u_n|\leq k\}}b_n$ is uniformly bounded, then we can use the Lebesgue dominated convergence theorem to get

$$\int_{\{|u_n|\leq k\}} b_n\varphi_n dx = \eta^2(n).$$

Therefore, one deduces from (4.90) that

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla\varphi_n dx + \int_\Omega g(x, u_n, \nabla u_n)\varphi_n dx \leq \eta^3(n). \tag{4.92}$$

Reasoning similarly as in the proof of Proposition 4.8, one deduces that

$$\lim_{n\to\infty} \int_\Omega [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))] \cdot \nabla(T_k(u_n) - T_k(u))dx = 0. \tag{4.93}$$

Using Lemma 3.3, one has

$$T_k(u_n) \to T_k(u) \text{ in } W^{1,p(\cdot)}(\Omega),$$

thus,

$$\nabla u_n \to \nabla u \text{ a.e. in } \Omega.$$

One also deduces that

$$a(x, u_n, \nabla u_n) \to a(x, u, \nabla u) \text{ a.e. in } \Omega$$

and

$$g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u) \text{ a.e. in } \Omega.$$

Since the sequence $(a(x, u_n, \nabla u_n))_{n\in\mathbb{N}}$ is bounded in $(L^{p'(\cdot)}(\Omega))^N$, one has

$$a(x, u_n, \nabla u_n) \rightharpoonup a(x, u, \nabla u) \text{ weakly in } (L^{p'(\cdot)}(\Omega))^N.$$

□

Proposition 4.10. *If u_n is a solution of $(\mathcal{P}^g_{f_n,\beta})$, then*

$$g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u) \text{ strongly in } L^1(\Omega). \tag{4.94}$$

Proof. Since $g_n(x,u_n,\nabla u_n) \longrightarrow g(x,u,\nabla u)$ a.e. in Ω as $n\to\infty$, thanks to (3.10), it suffice to prove that $g(x,u_n,\nabla u_n)$ is uniformly equi-integrable.
For any measurable subset E of Ω and for any $m\in\mathbb{R}^+$, one has

$$\begin{aligned}
\int_E |g_n(x,u_n,\nabla u_n)|dx &= \int_{E\cap\{|u_n|\le m\}} |g_n(x,u_n,\nabla u_n)|dx + \int_{E\cap\{|u_n|>m\}} |g_n(x,u_n,\nabla u_n)|dx\\
&= \int_{E\cap\{|u_n|\le m\}} |g_n(x,T_m(u_n),DT_m(u_n))|dx + \int_{E\cap\{|u_n|>m\}} |g_n(x,u_n,\nabla u_n)|dx\\
&\le b(m)\int_E (C(x)+|\nabla T_m(u_n)|^{p(x)})dx + \int_{E\cap\{|u_n|>m\}} |g_n(x,u_n,\nabla u_n)|dx\\
&= L_1+L_2. \qquad (4.95)
\end{aligned}$$

For fixed m in $\mathbb{N}$, one has

$$L_1 \le b(m)\int_E (C(x)+|\nabla T_m(u_n)|^{p(x)})dx,$$

which is small, uniformly in ϵ for fixed m when the measure of E is small (remark that $(\nabla T_m(u_n))_{n\in\mathbb{N}}$ converges strongly in $(L^{p(\cdot)}(\Omega))^N$ as $n\to\infty$).
We treat the second term of (4.95) by using the test function $S_m(u_n)$ in (4.72), where for $m>1$, S_m is defined as follows.

$$\begin{cases} S_m(s)=0, \ \text{ if } |s|\le m-1,\\[2mm] S_m(s)=\frac{|s|}{s}, \text{ if } |s|\ge m,\\[2mm] S'_m(s)=1, \quad \text{if } m-1\le|s|\le m.\end{cases}$$

It follows that

$$\int_\Omega b_n S_m(u_n)dx + \int_\Omega a(x,u_n,\nabla u_n)\nabla u_n S'_m(u_n)dx + \int_\Omega g(x,u_n,\nabla u_n)S_m(u_n)dx$$
$$= \int_\Omega fS_m(u_n)dx.$$

Using fact that the function $\beta_n\circ T_{\frac{1}{n}}$ is nondecreasing, S_m is increasing on $\{m-1\le|u_n|\le m\}$ and g_n verifies the sign condition, one deduces that

$$\int_{\{m-1\le|u_n|\le m\}} a(x,u_n,\nabla u_n)\nabla u_n dx + \int_{\{|u_n|>m-1\}} |g(x,u_n,\nabla u_n)|dx$$
$$\le \int_{\{|u_n|>m-1\}} |f_n|dx.$$

Using (3.5), one obtains

$$\begin{cases} \lambda\displaystyle\int_{\{m-1\le|u_n|\le m\}} |\nabla u_n|^{p(x)}dx + \int_{\{|u_n|>m-1\}} |g(x,u_n,\nabla u_n)|dx\\[2mm] \le \displaystyle\int_{\{m-1\le|u_n|\le m\}} a(x,u_n,\nabla u_n)\nabla u_n dx + \int_{\{|u_n|>m-1\}} |g(x,u_n,\nabla u_n)|dx\\[2mm] \le \displaystyle\int_{\{|u_n|>m-1\}} |f_n|dx. \end{cases} \qquad (4.96)$$

It follows that

$$\int_{\{|u_n|>m-1\}} |g(x,u_n,\nabla u_n)|dx \leq \int_{\{|u_n|\geq m-1\}} |f|dx,$$

then

$$\limsup_{n\to\infty} \int_{\{|u_n|>m-1\}} |g(x,u_n,\nabla u_n)|dx \leq \int_{\{|u|>m-1\}} |f|dx.$$

Thus, L_2 is also small, uniformly in n and in E when m is sufficiently large and one deduces that $g_n(x,u_n,\nabla u_n)$ is uniformly equi-integrable in Ω.
In view of Vitali's theorem, one has

$$g_n(x,u_n,\nabla u_n) \to g(x,u,\nabla u) \text{ strongly in } L^1(\Omega).$$

□

Thanks to the above convergences results, one can pass to the limit in (4.72) as $n\to\infty$ to obtain

$$\int_\Omega b\varphi dx + \int_\Omega a(x,u,\nabla u)\nabla\varphi dx + \int_\Omega g(x,u,\nabla u)\varphi dx = \int_\Omega f\varphi dx,$$

for any $\varphi \in W^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$. □

5. EXISTENCE OF ENTROPY AND RENORMALIZED SOLUTION

5.1. Existence of Entropy solution.

Here, we present our second main result. We prove the existence of entropy solutions of problem $(\mathcal{P}^g_{f,\beta})$ for any datum f in $L^1(\Omega)$.

Theorem 5.5. *For $f\in L^1(\Omega)$, there exists at least one entropy solution $(u,b)\in\mathcal{T}^{1,p(\cdot)}(\Omega)\times L^1(\Omega)$ of problem $(\mathcal{P}^g_{f,\beta})$ in the sense that $b(x)\in\beta(u(x))$ a.e. in Ω, $g(x,u,\nabla u)\in L^1(\Omega)$ and*

$$\int_\Omega bT_k(u-v)dx + \int_\Omega a(x,u,\nabla u)DT_k(u-v)dx + \int_\Omega g(x,u,\nabla u)T_k(u-v)dx$$

$$\leq \int_\Omega fT_k(u-v)dx, \tag{5.97}$$

for any $v\in W^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$.

Proof. Let us consider the approximating problem $(\mathcal{P}^g_{f_n,\beta})$ and its sequence of solutions (u_n,b_n) defined as in Section 5. For any $v\in W^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$, taking $T_k(u_n-v)$ as test function in (4.72) and setting $M=k+\|v\|_\infty$, one obtains

$$\int_\Omega b_nT_k(u_n-v)dx + \int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-v)dx + \int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n-v)dx$$

$$= \int_\Omega f_nT_k(u_n-v)dx. \tag{5.98}$$

Notice that if $|u_n| \geq M$, then $|u_n - v| \geq |u_n| - \|v\|_\infty > k$. Therefore, $\{|u_n - v| \leq k\} \subseteq \{|u_n| \leq M\}$, which gives

$$\begin{cases} \displaystyle\int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n - v)dx \\ \displaystyle= \int_\Omega a(x, T_M(u_n), \nabla T_M(u_n))(\nabla T_M(u_n) - \nabla v)\chi_{\{u_n - v| \leq k\}}dx \\ \displaystyle= \int_\Omega (a(x, T_M(u_n), \nabla T_M(u_n) - a(x, T_M(u_n), \nabla v))(\nabla T_M(u_n) - \nabla v)\chi_{\{u_n - v| \leq k\}}dx \\ \displaystyle+ \int_\Omega a(x, T_M(u_n), \nabla v)(\nabla T_M(u_n) - \nabla v)\chi_{\{u_n - v| \leq k\}}dx. \end{cases}$$

Using Fatou's Lemma, one gets

$$\begin{cases} \displaystyle\liminf_{n\to\infty} \int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n - v)dx \\ \displaystyle\geq \int_\Omega (a(x, T_M(u), \nabla T_M(u) - a(x, T_M(u), \nabla v))(\nabla T_M(u) - \nabla v)\chi_{\{u - v| \leq k\}}dx \\ \displaystyle+ \lim_{n\to\infty} \int_\Omega a(x, T_M(u_n), \nabla v)(\nabla T_M(u_n) - \nabla v)\chi_{\{u_n - v| \leq k\}}dx. \end{cases} \tag{5.99}$$

Since

$$\begin{cases} \displaystyle\lim_{n\to\infty} \int_\Omega a(x, T_M(u_n), \nabla v)(\nabla T_M(u_n) - \nabla v)\chi_{\{u_n - v| \leq k\}}dx \\ \displaystyle= \int_\Omega a(x, T_M(u), \nabla v)(\nabla T_M(u) - \nabla v)\chi_{\{u - v| \leq k\}}dx, \end{cases}$$

one deduces from (5.99), that

$$\begin{cases} \displaystyle\liminf_{n\to\infty} \int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n - v)dx \\ \displaystyle\geq \int_\Omega a(x, T_M(u), \nabla v)(\nabla T_M(u) - \nabla v)\chi_{\{u - v| \leq k\}}dx \\ \displaystyle= \int_\Omega a(x, T_M(u), \nabla v)(\nabla T_M(u) - \nabla v)\chi_{\{u - v| \leq k\}}dx \\ \displaystyle= \int_\Omega a(x, T_M(u), \nabla v)\nabla T_k(u - v)dx. \end{cases}$$

Since $T_k(u_n - v) \overset{*}{\rightharpoonup} T_k(u - v)$ in $L^\infty(\Omega)$ and $g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u)$ in $L^1(\Omega)$ as $n \to \infty$, one deduces that

$$\int_\Omega g_n(x, u_n, \nabla u_n)T_k(u_n - v)dx \longrightarrow \int_\Omega g(x, u, \nabla u)T_k(u - v)dx. \tag{5.100}$$

By using Lebesgue dominated convergence theorem, one obtains

$$\int_\Omega f_n T_k(u_n - v)dx \to \int_\Omega f T_k(u - v)dx. \tag{5.101}$$

Since $b_n \rightharpoonup b$ weakly in $L^1(\Omega)$ and $T_k(u_n - v) \overset{*}{\rightharpoonup} T_k(u-v)$ in $L^\infty(\Omega)$ as $n \to \infty$, it follows that

$$\int_\Omega b_n T_k(u_n - v)dx \to \int_\Omega bT_k(u-v)dx. \tag{5.102}$$

By passing to the limit in (4.72) as $n \to \infty$, one obtains the entropy inequality (5.97). □

5.2. **Existence of renormalized solution.**

In this section, we prove that an entropy solution is also a renormalized solution of problem $(\mathcal{P}^g_{f,\beta})$.

Theorem 5.6. *For $f \in L^1(\Omega)$, there exists at least one renormalized solution $(u,b) \in W^{1,p(.)}(\Omega) \times L^1(\Omega)$ of problem $(\mathcal{P}^g_{f,\beta})$ in the sense that $b(x) \in \beta(u(x))$ a.e. in Ω, $g(x,u,\nabla u) \in L^1(\Omega)$,*

$$\int_\Omega bS(u)vdx + \int_\Omega a(x,u,\nabla u)(S'(u)v\nabla u + S(u)\nabla v)dx + \int_\Omega g(x,u,\nabla u)S(u)vdx$$

$$= \int_\Omega fS(u)vdx \tag{5.103}$$

and

$$\lim_{l\to+\infty} \int_{\{l \le |u| \le l+1\}} |\nabla u|^{p(x)} dx = 0, \tag{5.104}$$

for any $v \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$ and for any smooth function $S \in W^{1,\infty}(\Omega)$ with compact support.

Proof. Let us start by proving that (5.104) holds.

Let u be an entropy solution of $(\mathcal{P}^g_{f,\beta})$. According to (4.96), one has

$$\lambda \int_{\{l \le |u_n| \le l+1\}} |\nabla u_n|^{p(x)} dx + \int_{\{|u_n|>l\}} |g(x,u_n,\nabla u_n)|dx \le \int_{\{|u_n|>l\}} |f_n|dx,$$

thus,

$$\lambda \int_{\{l \le |u_n| \le l+1\}} |\nabla u_n|^{p(x)} dx \le \int_{\{|u_n|>l\}} |f_n|dx.$$

Then, letting $n \to \infty$ and $l \to \infty$ successively, one obtains (5.104).

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of weak solution of (4.72) and $S \in W^{1,\infty}(\Omega)$ such that supp$S \subset [-M,M]$ for some $M > 0$. We already know from Lemma 4.85 that

for any $k > 0$, $T_k(u_n) \to T_k(u)$ strongly in $W^{1,p(.)}(\Omega)$ as $n \to \infty$.

For any $v \in C_0^\infty(\Omega)$, we choose $S(u_n)v \in W^{1,p(.)}(\Omega)$ as test function in (4.72) to obtain

$$\int_\Omega b_n S(u_n)vdx + \int_\Omega a(x,u_n,\nabla u_n)(S'(u_n)v\nabla u_n + S(u_n)\nabla v)dx$$

$$+ \int_\Omega g_n(x,u_n,\nabla u_n)S(u_n)vdx = \int_\Omega f_n S(u_n)vdx. \tag{5.105}$$

Since $S(u_n)v \overset{*}{\rightharpoonup} S(u)v$ in $L^\infty(\Omega)$ as $n \to \infty$, one deduces that

$$\int_\Omega b_n S(u_n)vdx \to \int_\Omega bS(u)vdx, \tag{5.106}$$

$$\int_\Omega g(x,u_n,\nabla u_n)S(u_n)vdx \to \int_\Omega g(x,u,\nabla u)S(u)vdx \tag{5.107}$$

and

$$\int_\Omega f_n S(u_n) v dx \to \int_\Omega f S(u) v dx. \tag{5.108}$$

It remain to treat the second term of the left on side of (5.105).
Indeed, one has

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)(S'(u_n)v\nabla u_n + S(u_n)\nabla v)dx\\ &= \int_\Omega a(x,T_M(u_n),\nabla T_M(u_n))(S'(u_n)v\nabla T_M(u_n) + S(u_n)\nabla v)dx.\end{aligned}$$

Thanks to (3.6), $(a(x,T_M(u_n),\nabla T_M(u_n)))_{n\in\mathbb{N}}$ is bounded in $(L^{p'(\cdot)}(\Omega))^N$ and

$$a(x,T_M(u_n),\nabla T_M(u_n)) \to a(x,T_M(u),\nabla T_M(u)) \text{ a.e. in } \Omega.$$

Then, using Lemma 2.2, one obtains

$$a(x,T_M(u_n),\nabla T_M(u_n)) \rightharpoonup a(x,T_M(u),\nabla T_M(u)) \text{ weakly in } (L^{p'(\cdot)}(\Omega))^N.$$

As $n\to\infty$, one has

$$(S'(u_n)v\nabla T_M(u_n) + S(u_n)\nabla v) \to (S'(u)v\nabla T_M(u) + S(u)\nabla v) \text{ strongly in } (L^{p'(\cdot)}(\Omega))^N,$$

then

$$\begin{aligned}&\lim_{n\to\infty}\int_\Omega a(x,T_M(u_n),\nabla T_M(u_n))(S'(u_n)v\nabla T_M(u_n) + S(u_n)\nabla v)dx\\ &= \int_\Omega a(x,T_M(u),\nabla T_M(u))(S'(u)v\nabla T_M(u) + S(u)\nabla v)dx\\ &= \int_\Omega a(x,u,\nabla u)(S'(u)v\nabla u + S(u)\nabla v)dx. \qquad (5.109)\end{aligned}$$

Since (5.106)-(5.109) hold, passing to the limit in (5.105) as $n\to\infty$, one obtains the renormalized equality (5.103), which is

$$\begin{aligned}\int_\Omega bS(u)vdx + \int_\Omega a(x,u,\nabla u)(S'(u)v\nabla u + S(u)\nabla v)dx + \int_\Omega g(x,u,\nabla u)S(u)vdx\\ = \int_\Omega fS(u)vdx.\end{aligned}$$

□

6. Conclusion

In the present article, we investigated the existence of weak and renormalized solutions of a class of multivalued Neumann boundary problem governed by the general $p(.)$-Leray-Lions type operator and involving a natural growth term and L^1 data. We also established that a renormalized solution coincides with an entropy one. Resulting equations are solved by approximation method and the technic of monotone operators in Banach spaces. This contribution provides important arguments whose permit to control the natural growth term. As far as the previous literature is concerned, a big step has already been taken (see [3]) in the study of this problem in the non-homogeneous Dirichlet framework. The main feature of this work is the fact that its extends the study of nonlinear elliptic problems into the framework of maximal monotone graph under Neumann boundary condition in variable exponent spaces.

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