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Forgotten Index of F^* Sums of Graphs

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ABSTRACT. The forgotten index is a degree-based topological index that was rediscovered in 2015 after being largely ignored for decades. In a recent paper, Alex *et al.* [5] computed the Zagreb indices of F^* sums. In this paper, we determine the forgotten index of F^* sums in terms of the forgotten indices of their component graphs. We also verified our results using basic graphs.

1. INTRODUCTION

Topological indices are numerical quantities associated with graphs that provide structural information about graphs and are preserved under graph isomorphisms. In 1972, I. Gutman and N. Trinajstić introduced the first degree-based topological index [16]. Later, in 1975 Gutman *et al.* [15] introduced two more degree based topological index to study the physical properties of compounds. The first index, which is now known as the second Zagreb index, was studied extensively in the following years. However, the second index, which is now known as the forgotten index, was largely ignored. In 2015, B. Furtula and I. Gutman rediscovered the forgotten index [13] and gave it its current name. They also studied its structural relationship and mathematical properties. Since then, numerous studies have been carried out on the forgotten index, and it has been shown to be correlated with a number of physical and chemical properties of molecules [13].

Let $G = (V(G), E(G))$ be a graph, then the first Zagreb index ($M_1(G)$), second Zagreb index ($M_2(G)$) and forgotten index ($F(G)$) are defined as follows:

$$M_1(G) = \sum_{u \in V(G)} d_G(u)^2 = \sum_{uv \in E(G)} (d_G(u) + d_G(v))$$

$$M_2(G) = \sum_{uv \in E(G)} (d_G(u) d_G(v))$$

$$F(G) = \sum_{u \in V(G)} d_G(u)^3 = \sum_{uv \in E(G)} (d_G(u)^2 + d_G(v)^2)$$

Where $d_G(u)$ denote the degree of the verex u in G. For more detailed accounts of the forgotten index, refer [1, 4, 6, 9, 14]. In 2015, a general version of Zagreb index called generalized Zagreb indices ($M_\alpha(G), \alpha > 3$) of graphs were proposed by X. Li and H. Zhao [17, 18]. For a graph $G = (V(G), E(G))$, the generalized Zagreb index is defined as follows:

$$M_\alpha(G) = \sum_{u \in V(G)} d_G(u)^\alpha$$

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The study of graph operations is a fundamental problem in chemical graph theory. Recently, several topological indices of various graph operations have been computed [2, 3, 4, 8, 19]. Yan *et al.* computed the Wiener index of four subdivision graphs $S(G), R(G), Q(G), T(G)$ proposed by Cvetković *et al.* [10, 20]. In 2009, M. Eliasi and B. Taeri proposed a new operation called the F sum, which is associated with subdivision graphs in terms of the Cartesian product, and computed the Wiener index of the F sum [12]. Hanyuan Deng *et al.* computed the Zagreb indices of F sums [11] in terms of its constituent graphs. Shehnaz Akhtar *et al.* computed the forgotten index of F sums [1]. In 2019, JB Liu *et al.* defined the generalized form of subdivisions namely $\{S_k, R_k, Q_k, T_k\}$ and computed the first and second Zagreb index of the associated F_k sums [19]. HM Awais *et al.* determined the forgotten index of the genearalized F sums [6].

Several new graph operations associated with subdivision graphs have been proposed recently. One of them is the F^* sums [5] proposed by the authors. We also computed the first and second Zagreb indices of F^* sums. In this paper, we compute the forgotten index of F^* sums in terms of its component graphs and illustrate the expressions with some examples. Throughout this paper we consider only simple, finite, undirected and connected graphs. The following sections are arranged as follows: Section 2 contains preliminaries and basic definitions, Section 3 includes main results and illustations and Section 4 contains the conclusion.

2. PRELIMINARIES

Let $G_1 = (V_1, E_1)$ be a graph. The subdivison graphs associated with G_1 is defined as [10, 20] follows.

(1) $S(G_1)$ is the graph obtained from G_1 by replacing each edge e_i of G_1 by a path of length 2. The collection of new vertices in $S(G_1)$ is denoted by V_1^*. Thus the vertex set $V(S(G_1)) = V_1 \cup V_1^*$ and the edge set $E(S(G_1)) = \{(v, h), (u, h) : e = vu \in E_1, h \in V_1^*\}$. $S(G)$ is also known as edge subdivision graph.
(2) $R(G_1)$ is the graph obtained from G_1 by replacing each edge e_i of G_1 by a path of length 2 and keeping the original edges of G_1. That is the vertex set of $R(G_1)$ is $V(R(G_1)) = V_1 \cup V_1^*$ and edge set $E(R(G_1)) = E(S(G_1)) \cup E_1$.
(3) $Q(G_1)$ is the graph obtained from G_1 by replacing each edge e_i of G_1 by a path of length 2 and making new vertices adjacent whenever the corresponding edges are adjacent in G_1. That is, $Q(G_1)$ is a graph with vertex set $V(Q(G_1)) = V_1 \cup V_1^*$ and edge set $E(Q(G_1)) = \{(v, h), (u, h) : e = vu \in E_1, h \in V_1^*\} \cup E_1^*$, $E_1^* = \{(u_i, u_j) : e_i \text{ adjacent to } e_j \ in\ E_1, u_i, u_j \in V_1^*\}$ where u_i, u_j are the vertices corresponding to the edges $e_i, e_j \in E_1$.
(4) $T(G_1)$ is the graph obtained from G_1 by replacing each edge e_i of G_1 by a path of length 2 and making new vertices adjacent whenever the corresponding edges are adjacent in G_1 also keeping the original edges in G_1. That is, $T(G_1)$ is a graph with vertex set $V(T(G_1)) = V_1 \cup V_1^*$ and edge set $E(T(G_1)) = E(Q(G_1)) \cup E_1$.

Associated with these graphs we have F^* sum defined as

Definition 2.1. [5] *Let F be any one of the symbols S, R, Q, T, then the F^* sum of two graphs G_1 and G_2 is denoted by $G_1 *_F G_2$, is the graph with vertex set $V(G_1 *_F G_2) = V(F(G_1)) \times V_2$ and the edge set $E(G_1 *_F G_2) = \{(a, b)(c, d) : a = c \in V_1^*$ and $bd \in E_2$ or $ac \in E(F(G_1))$ and $b = d \in V_2\}$.*

For example, consider Figure 2 in which $G_1 = P_4$ and $G_2 = P_5$.

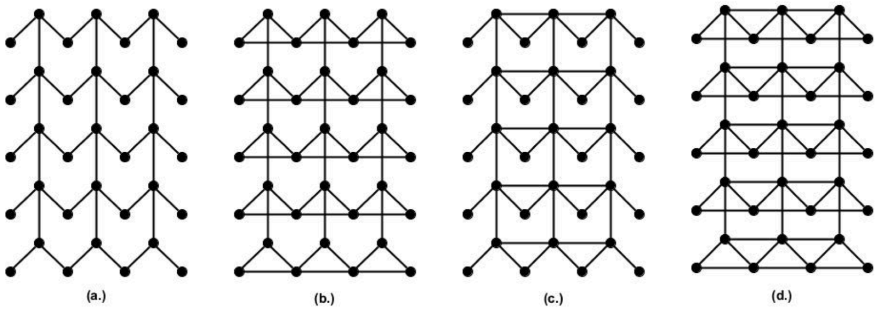

FIGURE 1. $(a.)P_4 *_S P_5 (b.)P_4 *_R P_5 (c.)P_4 *_Q P_5 (d.)P_4 *_T P_5$

3. FORGOTTEN INDEX OF F^* SUM

In this section we compute the expression for forgotten index of F^* sums of graphs.

Theorem 3.1. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two connected graphs. Then*

$$F(G_1 *_S G_2) = |V_2|F(G_1) + |E_1|F(G_2) + 6|E_1|M_1(G_2) + 24|E_1||E_2| + 8|E_1||V_2|$$

Proof. From the definition of forgotten index, we have

$$\begin{aligned}
F(G_1 *_S G_2) &= \sum_{(u,v)\in V(G_1 *_S G_2)} (d_{(G_1*_S G_2)}(u,v))^3 \\
&= \sum_{(u,v)(x,y)\in E(G_1*_S G_2)} \left((d_{(G_1*_S G_2)}(u,v))^2 + (d_{(G_1*_S G_2)}(x,y))^2\right) \\
&= \sum_{u\in V_1^*} \sum_{vy\in E_2} \left((d_{(G_1*_S G_2)}(u,v))^2 + (d_{(G_1*_S G_2)}(u,y))^2\right) \\
&\quad + \sum_{v\in V_2} \sum_{ux\in E(S(G_1))} \left((d_{(G_1*_S G_2)}(u,v))^2 + (d_{(G_1*_S G_2)}(x,v))^2\right)
\end{aligned}$$

Now we separately find the values of the each parts in the sum. Firstly we consider the sum in which $u \in V_1^*$ and $vy \in E_2$.

$$\begin{aligned}
&\sum_{u\in V_1^*} \sum_{vy\in E_2} \left((d_{(G_1*_S G_2)}(u,v))^2 + (d_{(G_1*_S G_2)}(u,y))^2\right) \\
&= \sum_{u\in V_1^*} \sum_{vy\in E_2} \left[(d_{S(G_1)}(u) + d_{G_2}(v))^2 + \left(d_{S(G_1)}(u) + d_{G_2}(y)\right)^2\right] \\
&= \sum_{u\in V_1^*} \sum_{vy\in E_2} \left[2(d_{S(G_1)}(u))^2 + (d_{G_2}(v))^2 + (d_{G_2}(y))^2 + 2d_{S(G_1)}(u)\left(d_{G_2}(v) + d_{G_2}(y)\right)\right] \\
&= \sum_{u\in V_1^*} [8|E_2| + F(G_2) + 4M_1(G_2)] = 8|E_1||E_2| + |E_1|F(G_2) + 4|E_1|M_1(G_2)
\end{aligned}$$

Now for each edge $ux \in E(S(G_1))$, $v \in V_2$.

$$\begin{aligned}
&\sum_{v\in V_2}\sum_{ux\in E(S(G_1))}\left((d_{(G_1*_SG_2)}(u,v))^2+(d_{(G_1*_SG_2)}(x,v))^2\right)\\
&=\sum_{v\in V_2}\sum_{\substack{ux\in E(S(G_1))\\ u\in V_1,x\in V_1^*}}\left[(d_{G_1}(u))^2+\left(d_{G_2}(v)+d_{S(G_1)}(x)\right)^2\right]\\
&=\sum_{v\in V_2}\left(F(G_1)+2|E_1|(d_{G_2}(v))^2+8|E_1|d_{G_2}(v)+8|E_1|\right)\\
&=|V_2|F(G_1)+2|E_1|M_1(G_2)+16|E_1||E_2|+8|E_1||V_2|
\end{aligned}$$

From the expressions, we obtain

$$\begin{aligned}
F(G_1*_S G_2)=&|V_2|F(G_1)+|E_1|F(G_2)+6|E_1|M_1(G_2)\\
&+24|E_1||E_2|+8|E_1||V_2|
\end{aligned}$$

□

Theorem 3.2. *Let $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ be two connected graphs. Then*

$$F(G_1*_R G_2)=8|V_2|F(G_1)+|E_1|F(G_2)+6|E_1|M_1(G_2)+24|E_1||E_2|+8|E_1||V_2|$$

Proof. We have,

$$\begin{aligned}
F(G_1*_R G_2)&=\sum_{(u,v)\in V(G_1*_RG_2)}(d_{(G_1*_RG_2)}(u,v))^3\\
&=\sum_{(u,v)(x,y)\in E(G_1*_RG_2)}\left((d_{(G_1*_RG_2)}(u,v))^2+(d_{(G_1*_RG_2)}(x,y))^2\right)\\
&=\sum_{u\in V_1^*}\sum_{vy\in E_2}\left((d_{(G_1*_RG_2)}(u,v))^2+(d_{(G_1*_RG_2)}(u,y))^2\right)\\
&+\sum_{v\in V_2}\sum_{ux\in E(R(G_1))}\left((d_{(G_1*_RG_2)}(u,v))^2+(d_{(G_1*_RG_2)}(x,v))^2\right)
\end{aligned}$$

Now we separately find the values of the each parts in the sum. Firstly we consider the sum in which $u \in V_1^*$ and $vy \in E_2$.

$$\begin{aligned}
&\sum_{u\in V_1^*}\sum_{vy\in E_2}\left((d_{(G_1*_RG_2)}(u,v))^2+(d_{(G_1*_RG_2)}(u,y))^2\right)\\
&=\sum_{u\in V_1^*}\sum_{vy\in E_2}\left[\left(d_{R(G_1)}(u)+d_{G_2}(v)\right)^2+\left(d_{R(G_1)}(u)+d_{G_2}(y)\right)^2\right]\\
&=\sum_{u\in V_1^*}\sum_{vy\in E_2}\left[8+4(d_{G_2}(v)+d_{G_2}(y))+(d_{G_2}(v))^2+(d_{G_2}(y))^2\right]\\
&=\sum_{u\in V_1^*}[8|E_2|+4M_1(G_2)+F(G_2)]=8|E_1||E_2|+4|E_1|M_1(G_2)+|E_1|F(G_2)
\end{aligned}$$

Now for each edge $ux \in E(R(G_1))$, $v \in V_2$.

$$\sum_{v\in V_2}\sum_{ux\in E(R(G_1))}\left((d_{(G_1*_RG_2)}(u,v))^2+(d_{(G_1*_RG_2)}(x,v))^2\right)$$
$$=\sum_{v\in V_2}\sum_{ux\in E(R(G_1)),u,x\in V_1}\left((d_{(G_1*_RG_2)}(u,v))^2+(d_{(G_1*_RG_2)}(x,v))^2\right)$$
$$+\sum_{v\in V_2}\sum_{ux\in E(R(G_1)),u\in V_1,x\in V_1^*}\left((d_{(G_1*_RG_2)}(u,v))^2+(d_{(G_1*_RG_2)}(x,v))^2\right)$$

Now we calculate the expressions separately

$$\sum_{v\in V_2}\sum_{ux\in E(R(G_1)),u,x\in V_1}\left((d_{(G_1*_RG_2)}(u,v))^2+(d_{(G_1*_RG_2)}(x,v))^2\right)$$
$$=\sum_{v\in V_2}\sum_{ux\in E(R(G_1))u,x\in V_1}\left((d_{R(G_1)}(u))^2+(d_{R(G_1)}(x))^2\right)$$
$$=\sum_{v\in V_2}\sum_{ux\in E(R(G_1))u,x\in V_1}4\left((d_{G_1}(u))^2+(d_{G_1}(x))^2\right)$$
$$=4|V_2|F(G_1)$$

Also, consider the case where $ux \in E(R(G_1)), u \in V_1, x \in V_1^*$

$$\sum_{v\in V_2}\sum_{ux\in E(R(G_1)),u\in V_1,x\in V_1^*}\left((d_{(G_1*_RG_2)}(u,v))^2+(d_{(G_1*_RG_2)}(x,v))^2\right)$$
$$=\sum_{v\in V_2}\sum_{ux\in E(R(G_1)),u\in V_1,x\in V_1^*}(d_{R(G_1)}(u))^2+(d_{G_2}(v)\ +2)^2$$
$$=\sum_{v\in V_2}\sum_{ux\in E(R(G_1)),u\in V_1,x\in V_1^*}\left(4(d_{G_1}(u))^2+(d_{G_2}(v))^2+4d_{G_2}(v)+4\right)$$
$$=4|V_2|F(G_1)+2|E_1|M_1(G_2)+16|E_1||E_2|+8|E_1||V_2|$$

From the expressions, we obtain

$$F(G_1*_RG_2)=8|V_2|F(G_1)+|E_1|F(G_2)+6|E_1|M_1(G_2)+24|E_1||E_2|+8|E_1||V_2|$$

□

Theorem 3.3. [11] *Let $G_1=(V_1,E_1)$ be a connected graph. Then*

$$\sum_{u_iu_j,u_ju_k\in E_1}(d_{G_1}(u_i)+d_{G_1}(u_j)+d_{G_1}(u_j)+d_{G_1}(u_k))=2\sum_{u_j\in V_1}C^2_{d_{G_1}(u_j)}d_{G_1}(u_j)$$
$$+\sum_{u_j\in V_1}(d_{G_1}(u_j)-1)\sum_{u_i\in V_1,u_iu_j\in E_1}d_{G_1}(u_i)$$

Theorem 3.4. [1] *Let $G_1=(V_1,E_1)$ be a connected graph. Then*

$$\sum_{u_iu_j,u_ju_k\in E_1}\left((d_{G_1}(u_i)+d_{G_1}(u_j))^2+(d_{G_1}(u_j)+d_{G_1}(u_k))^2\right)$$
$$=\sum_{u_j\in V_1}C^2_{d_{G_1}(u_j)}(d_{G_1}(u_j))^2+\sum_{u_j\in V_1}(d_{G_1}(u_j)-1)\sum_{u_i\in V_1,u_iu_j\in E_1}(d_{G_1}(u_i))^2$$
$$+\sum_{u_j\in V_1}d_{G_1}(u_j)(d_{G_1}(u_j)-1)\sum_{u_i\in V_1,u_iu_j\in E_1}d_{G_1}(u_i)$$

Theorem 3.5. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two connected graphs. Then*

$$\begin{aligned}
F(G_1 *_Q G_2) = {} & |V_2|M_4(G_1) + (6|E_2| + |V_2|)F(G_1) + |E_1|F(G_2) + (8|E_2| - 2|V_2|)M_2(G_1) \\
& - 4|E_2|M_1(G_1) + 2M_1(G_1)M_1(G_2) + 2(|E(Q(G_1))| - 2|E_1|)M_1(G_2) \\
& + |V_2| \left(\sum_{u_i \in V_1} \left(\sum_{u_j \in N_{G_1}(u_i)} d_{G_1}(u_j)(d_{G_1}(u_j) - 1) \right) \right) \\
& + |V_2| \left(\sum_{u_i u_j \in E_1} (d_{G_1}(u_i) + d_{G_1}(u_j)) d_{G_1}(u_i) d_{G_1}(u_j) \right)
\end{aligned}$$

Proof. We have,

$$\begin{aligned}
F(G_1 *_Q G_2) &= \sum_{(u,v) \in V(G_1 *_Q G_2)} (d_{(G_1 *_Q G_2)}(u,v))^3 \\
&= \sum_{(u,v)(x,y) \in E(G_1 *_Q G_2)} \left((d_{(G_1 *_Q G_2)}(u,v))^2 + (d_{(G_1 *_Q G_2)}(x,y))^2 \right) \\
&= \sum_{u \in V_1^*} \sum_{vy \in E_2} \left((d_{(G_1 *_Q G_2)}(u,v))^2 + (d_{(G_1 *_Q G_2)}(u,y))^2 \right) \\
&+ \sum_{v \in V_2} \sum_{ux \in E(Q(G_1))} \left((d_{(G_1 *_Q G_2)}(u,v))^2 + (d_{(G_1 *_Q G_2)}(x,v))^2 \right)
\end{aligned}$$

First we consider the sum in which $u \in V_1^*$ and $vy \in E_2$.

$$\begin{aligned}
& \sum_{u \in V_1^*} \sum_{vy \in E_2} \left((d_{(G_1 *_Q G_2)}(u,v))^2 + (d_{(G_1 *_Q G_2)}(u,y))^2 \right) \\
&= \sum_{u \in V_1^*} \sum_{vy \in E_2} \left[\left(d_{Q(G_1)}(u) + d_{G_2}(v) \right)^2 + \left(d_{Q(G_1)}(u) + d_{G_2}(y) \right)^2 \right] \\
&= \sum_{u \in V_1^*} \sum_{vy \in E_2} \left[2(d_{Q(G_1)}(u))^2 + 2d_{Q(G_1)}(u) \left(d_{G_2}(v) + d_{G_2}(y) \right) + (d_{G_2}(v))^2 + (d_{G_2}(y))^2 \right] \\
&= \sum_{e=pq \in E_1} 2|E_2|(d_{G_1}(p) + d_{G_1}(q))^2 + \sum_{e=pq \in E_1} 2(d_{G_1}(p) + d_{G_1}(q))M_1(G_2) + \sum_{u \in V_1^*} F(G_2) \\
&= 2|E_2|F(G_1) + 4|E_2|M_2(G_1) + 2M_1(G_1)M_1(G_2) + |E_1|F(G_2)
\end{aligned}$$

For each edge $ux \in E(Q(G_1))$ and the vertex $v \in V_2$

$$\begin{aligned}
& \sum_{v \in V_2} \sum_{ux \in E(Q(G_1))} \left((d_{(G_1 *_Q G_2)}(u,v))^2 + (d_{(G_1 *_Q G_2)}(x,v))^2 \right) \\
&= \sum_{v \in V_2} \sum_{ux \in E(Q(G_1)), u \in V_1, x \in V_1^*} \left((d_{(G_1 *_Q G_2)}(u,v))^2 + (d_{(G_1 *_Q G_2)}(x,v))^2 \right) \\
&+ \sum_{v \in V_2} \sum_{ux \in E(Q(G_1)), u, x \in V_1^*} \left((d_{(G_1 *_Q G_2)}(u,v))^2 + (d_{(G_1 *_Q G_2)}(x,v))^2 \right)
\end{aligned}$$

Now we separately find both the expressions. First,

$$\begin{aligned}
&\sum_{v\in V_2}\ \sum_{ux\in E(Q(G_1)),u\in V_1,x\in V_1^*} \big((d_{(G_1*_Q G_2)}(u,v))^2+(d_{(G_1*_Q G_2)}(x,v))^2\big)\\
&=\sum_{v\in V_2}\ \sum_{ux\in E(Q(G_1)),u\in V_1,x\in V_1^*} (d_{Q(G_1)}(u))^2+\big(d_{G_2}(v)+d_{Q(G_1)}(x)\big)^2\\
&=\sum_{v\in V_2}\ \sum_{ux\in E(Q(G_1)),u\in V_1,x\in V_1^*} (d_{G_1}(u))^2+(d_{G_2}(v)+d_{Q(G_1)}(x))^2\\
&=\sum_{v\in V_2}(F(G_1)+2|E_1|(d_{G_2}(v))^2)\\
&+\sum_{v\in V_2}\ \sum_{e=u_iv_i\in E(G_1),u_i,v_i\in V_1,} \big[2(d_{G_1}(u_i)+d_{G_1}(v_i))d_{G_2}(v)+(d_{G_1}(u_i)+d_{G_1}(v_i))^2\big]\\
&=|V_2|F(G_1)+2|E_1|M_1(G_2)+4|E_2|M_1(G_1)+|V_2|F(G_1)+2|V_2|M_2(G_1)
\end{aligned}$$

the second part of the sum is,

$$\begin{aligned}
&\sum_{v\in V_2}\ \sum_{ux\in E(Q(G_1)),u,x\in V_1^*} \big((d_{(G_1*_Q G_2)}(u,v))^2+(d_{(G_1*_Q G_2)}(x,v))^2\big)\\
&=\sum_{v\in V_2}\ \sum_{ux\in E(Q(G_1)),u,x\in V_1^*} \big((d_{Q(G_1)}(u)+d_{G_2}(v))^2+(d_{Q(G_1)}(x)+d_{G_2}(v))^2\big)\\
&=\sum_{v\in V_2}\left(\sum_{ux\in E(Q(G_1)),u,x\in V_1^*} 2(d_{G_2}(u))^2\right)\\
&+\sum_{v\in V_2}\left(\sum_{ux\in E(Q(G_1)),u,x\in V_1^*} 2d_{G_2}(v)\big(d_{Q(G_1)}(u)+d_{Q(G_1)}(x)\big)\right)\\
&+\sum_{v\in V_2}\left(\sum_{ux\in E(Q(G_1)),u,x\in V_1^*} \big((d_{Q(G_1)}(u))^2+(d_{Q(G_1)}(x))^2\big)\right)\\
&=\sum_{v\in V_2}\left(\sum_{ux\in E(Q(G_1)),u,x\in V_1^*} 2(d_{G_2}(v))^2\right)\\
&+\sum_{v\in V_2}\left(\sum_{u_iu_j,u_ju_k\in E_1} 2d_{G_2}(v)\left(d_{G_1}(u_i)+d_{G_1}(u_j)+d_{G_1}(u_j)+d_{G_1}(u_k)\right)\right)\\
&+\sum_{v\in V_2}\left(\sum_{u_iu_j,u_ju_k\in E_1} \big((d_{G_1}(u_i)+d_{G_1}(u_j))^2+(d_{G_1}(u_j)+d_{G_1}(u_k))^2\big)\right)
\end{aligned}$$

Where u_iu_j is the edge corresponding to the vertex u and u_ju_k is the edge corresponding to the vertex x. Now using the expressions in Theorem 3.3 and Theorem 3.4 we can obtain the following.

$$
\begin{aligned}
&= 2(|E(Q(G_1))| - 2|E_1|)M_1(G_2) \\
&+ 4|E_2| \left(2 \sum_{u_j \in V_1} C^2_{d_{G_1}(u_j)} d_{G_1}(u_j) + \sum_{u_j \in V_1} (d_{G_1}(u_j) - 1) \sum_{u_i \in V_1, u_i u_j \in E_1} d_{G_1}(u_i) \right) \\
&+ 2|V_2| \left(\sum_{u_j \in V_1} C^2_{d_{G_1}(u_j)} (d_{G_1}(u_j))^2 + \sum_{u_j \in V_1} (d_{G_1}(u_j) - 1) \sum_{u_i \in V_1, u_i u_j \in E_1} (d_{G_1}(u_i))^2 \right) \\
&+ 2|V_2| \left(\sum_{u_j \in V_1} d_{G_1}(u_j)(d_{G_1}(u_j) - 1) \sum_{u_i \in V_1, u_i u_j \in E_1} d_{G_1}(u_i) \right)
\end{aligned}
$$

$$
\begin{aligned}
&= 2(|E(Q(G_1))| - 2|E_1|)M_1(G_2) \\
&+ 4|E_2| \left(\sum_{u_j \in V_1} (d_{G_1}(u_j)^3 - d_{G_1}(u_j)^2) + \sum_{u_j \in V_1} (d_{G_1}(u_j) - 1) \sum_{u_i \in V_1, u_i u_j \in E_1} d_{G_1}(u_i) \right) \\
&+ |V_2| \left(\sum_{u_j \in V_1} (d_{G_1}(u_j)^4 - d_{G_1}(u_j)^3) + \sum_{u_j \in V_1} (d_{G_1}(u_j) - 1) \sum_{u_i \in V_1, u_i u_j \in E_1} (d_{G_1}(u_i))^2 \right) \\
&+ 2|V_2| \left(\sum_{u_j \in V_1} d_{G_1}(u_j)(d_{G_1}(u_j) - 1) \sum_{u_i \in V_1, u_i u_j \in E_1} d_{G_1}(u_i) \right) \\
&= 2(|E(Q(G_1))| - 2|E_1|)M_1(G_2) + 4|E_2| \left(F(G_1) + 2M_2(G_1) - 2M_1(G_1) \right) \\
&+ |V_2| \left(M_4(G_1) - F(G_1) - 4M_2(G_1) \right) \\
&+ |V_2| \left(\sum_{u_i \in V_1} \left(\sum_{u_j \in N_{G_1}(u_i)} d_{G_1}(u_j)(d_{G_1}(u_j) - 1) \right) + \sum_{u_i u_j \in E_1} (d_{G_1}(u_i) + d_{G_1}(u_j)) d_{G_1}(u_i) d_{G_1}(u_j) \right)
\end{aligned}
$$

Thus we obtain,

$$
\begin{aligned}
F(G_1 *_Q G_2) &= |V_2| M_4(G_1) + (6|E_2| + |V_2|) F(G_1) + |E_1| F(G_2) \\
&+ (8|E_2| - 2|V_2|) M_2(G_1) - 4|E_2| M_1(G_1) \\
&+ 2M_1(G_1) M_1(G_2) + 2(|E(Q(G_1))| - 2|E_1|) M_1(G_2) \\
&+ |V_2| \left(\sum_{u_i \in V_1} \left(\sum_{u_j \in N_{G_1}(u_i)} d_{G_1}(u_j)(d_{G_1}(u_j) - 1) \right) \right) \\
&+ |V_2| \left(\sum_{u_i u_j \in E_1} (d_{G_1}(u_i) + d_{G_1}(u_j)) d_{G_1}(u_i) d_{G_1}(u_j) \right)
\end{aligned}
$$

□

Using the results obtained in Theorem 3.2 and Theorem 3.5 we can obtain the following result.

Theorem 3.6. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two connected graphs. Then*

$$\begin{aligned}F(G_1 *_T G_2) = &|V_2|M_4(G_1) + (6|E_2| + 5|V_2|)F(G_1) + |E_1|F(G_2) + (8|E_2| - 2|V_2|)M_2(G_1)\\ &- 4|E_2|M_1(G_1) + 2M_1(G_1)M_1(G_2) + 2(|E(Q(G_1))| - 2|E_1|)M_1(G_2)\\ &+ |V_2|\left(\sum_{u_i\in V_1}\left(\sum_{u_j\in N_{G_1}(u_i)} d_{G_1}(u_j)(d_{G_1}(u_j) - 1)\right)\right)\\ &+ |V_2|\left(\sum_{u_iu_j\in E_1}(d_{G_1}(u_i) + d_{G_1}(u_j))d_{G_1}(u_i)d_{G_1}(u_j)\right)\end{aligned}$$

Proof. When $u \in V_1^*$ and $vy \in E_2$,

$$\begin{aligned}&\sum_{u\in V_1^*}\sum_{vy\in E_2}\left(d_{(G_1*_TG_2)}(u,v)\right)^2 + \left(d_{(G_1*_TG_2)}(u,y)\right)^2\\ &= \sum_{u\in V_1^*}\sum_{vy\in E_2}\left(d_{(G_1*_QG_2)}(u,v)\right)^2 + \left(d_{(G_1*_QG_2)}(u,y)\right)^2\end{aligned}$$

When $v \in V_1$ and $ux \in E(T(G_1))$,

$$\begin{aligned}&\sum_{v\in V_2}\sum_{ux\in E(T(G_1))}\left(d_{(G_1*_TG_2)}(u,v)\right)^2 + \left(d_{(G_1*_TG_2)}(x,v)\right)^2\\ &= \sum_{v\in V_2}\sum_{ux\in E(T(G_1)),u\in V_1,x\in V_1^*}\left(d_{(G_1*_QG_2)}(u,v)\right)^2 + \left(d_{(G_1*_QG_2)}(x,v)\right)^2\\ &+ \sum_{v\in V_2}\sum_{ux\in E(T(G_1)),u,x\in V_1^*}\left(d_{(G_1*_QG_2)}(u,v)\right)^2 + \left(d_{(G_1*_QG_2)}(x,v)\right)^2\\ &+ \sum_{v\in V_2}\sum_{ux\in E(T(G_1)),u,x\in V_1}\left(d_{(G_1*_RG_2)}(u,v)\right)^2 + \left(d_{(G_1*_RG_2)}(x,v)\right)^2\end{aligned}$$

Now, from Theorem 3.2 and Theorem 3.5 we get the required expression. □

The above computational procedure can be used to find the forgotten index for many classes of graphs. As an illustration we provide the following.

Example 3.1. *Let $G_1 = P_n$ and $G_2 = P_m$, $n, m, > 3$. Then*

a. $F(P_n *_S P_m) = 72mn - 78m - 74n + 74$
b. $F(P_n *_R P_m) = 128mn - 176m - 74n + 74$
c. $F(P_n *_Q P_m) = 224mn - 412m - 182n + 302$
d. $F(P_n *_T P_m) = 280mn - 510m - 182n + 302$

Example 3.2. *Let $G_1 = C_n$ and $G_2 = C_m, n, m > 3$. Then*

a. $F(C_n *_S C_m) = 72mn$
b. $F(C_n *_R C_m) = 224mn$
c. $F(C_n *_Q C_m) = 128mn$
d. $F(C_n *_T C_m) = 280mn$

The same class of torus grid graph $\mathcal{T}_{n,m}$ [5] can be obtained from the given operation, $\mathcal{T}_{2n,m} = C_n *_S C_m$. Therefore, the forgotten index of $F(\mathcal{T}_{n,m}) = 72mn$.

4. Summary and Conclusion

In this paper, we have computed the forgotten index of F^* sums of graphs. Using the explicit expressions, we also computed the forgotten index of some structures as well. The computation of other topological indices of F^* sums is a problem for further research.

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