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Uniqueness of Shift Polynomials and Derivative Sharing Polynomial

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ABSTRACT. This research aims to establish a significant uniqueness theorem concerning shift polynomials and derivative-sharing polynomials connected to meromorphic functions. The study provides an example highlighting the constraints' significance in this context. The resulting theorem expands the existing literature under certain appropriate conditions.

1. PRELIMINARIES

In this paper, when we mention a meromorphic function, we are referring to a function that exhibits meromorphic behavior across the entire complex plane, as symbolized by $\mathbb{C}$. We assume that the reader is familiar with the conventional notations and important results of Nevanlinna theory regarding the value distribution of meromorphic functions(see [4]). Let $\mathfrak{D} = \{f : f$ is non-constant meromorphic function in $\mathbb{C}\}$. For the context of our study, we use the notation $n(r, \infty; f)$ to signify the count of poles of the function f within the region defined by $|z| < r$, with each pole taken into account along with its respective multiplicities.

Definition 1.1. [17]

$$N(r, \infty; f) = \int_0^r \frac{n(t, \infty; f) - n(0, \infty; f)}{t} dt + n(0, \infty; f) \log r$$

is referred to as the integrated counting function or, simply, the pole counting function of the function f.

The proximity function describing the poles of function f can be represented as follows $m(r, \infty; f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \mid f\left(re^{i\theta}\right) \mid d\theta$, where

$$\log^+ x = \begin{cases} x, & \text{if } x \geq 1, \\ 0, & \text{if } 0 \leq x < 1 \end{cases}$$

The Nevanlinna characteristic function $T(r, f)$ of f is expressed as the combination of two components: the quantity $m(r, \infty; f)$ and the integrated counting function $N(r, \infty; f)$, which together give rise to $T(r, f)$. We use the conventional notation $S(r, f)$ for any quantity that satisfies the relationship $\frac{S(r,f)}{T(r,f)} \to 0$ as r tends to infinity, with the exception of a finite set of linear measure.
For $a \in \mathbb{C}$, we write $m(r, a; f) = m\left(r, \infty; \frac{1}{f-a}\right)$ and $N(r, a; f) = N\left(r, \infty; \frac{1}{f-a}\right)$.
Again let us denote by $\overline{n}(r, a; f)$ the number of distinct a points of f lying in $\mid z \mid < r$, where $a \in \mathbb{C} \bigcup \{\infty\}$.

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Definition 1.2. [4] *Consider a positive integer, denoted as k, and an element a belonging to the set $\mathbb{C}\cup\{\infty\}$. We employ the notations $N_{k)}(r,a;f)$ and $N_{(k}(r,a;f)$ to represent the counting function of a-points in the function f with multiplicity not exceeding k and the counting function of a-points of f with multiplicity greater than k, respectively. Similarly, $\overline{N}_{k)}(r,a;f)$ and $\overline{N}_{(k}(r,a;f)$ denote the corresponding reduced functions. For $a \in \mathbb{C}$ and a positive integer p, we use the notation $N_p(r,a;f)$ to represent the sum $\overline{N}_{(1}(r,a;f)+\overline{N}_{(2}(r,a;f)+\cdots+\overline{N}_{(p}(r,a;f)$.*

Consider two non-constant meromorphic functions, denoted as $f(z)$ and $g(z)$. Let $a(z)$ be a small function concerning both $f(z)$ and $g(z)$. If the zeros of $f(z)-a(z)$ and $g(z)-a(z)$ are identical in number and multiplicity, we refer to $f(z)$ and $g(z)$ as sharing $a(z)$ with CM (Counting Multiplicities). When we disregard the multiplicities, we say that $f(z)$ and $g(z)$ share $a(z)$ with IM (Ignoring Multiplicities).
A noteworthy subtopic within the uniqueness theory is the shared values, functions, or sets between a meromorphic function and its derivative.

The study of difference equations and products in the complex plane has become a focal point for many mathematicians in recent years. Numerous publications have explored the value distribution of differences and difference operators within the context of Nevanlinna theory.

Rubel and Yang were the forefront leaders in the investigation of entire functions that exhibit shared values with their derivatives. In 1977, they established the following significant theorem.

Theorem 1. [12] *Let a and b be complex numbers such that $b \neq a$ and let $f(z)$ be a non-constant entire function. If $f(z)$ and $f'(z)$ share the values a and b CM, then $f \equiv f'$.*

Subsequently, this outcome has undergone numerous extensions and enhancements[18]. Recent studies have further expanded on these concepts. For instance, Priyanka V. et al. [11] examined the uniqueness of differential-difference polynomials of meromorphic functions sharing shift polynomial and small function. Tejuswini and Shilpa [16] investigated the unicity of shift polynomials generated by meromorphic functions.
In 1980, G. G. Gundersen enhanced Theorem 1 and derived the subsequent result.

Theorem 2. [3] *Let f be a non-constant meromorphic function, a and b be two distinct finite values. If f and f' share the values a and b CM, then $f \equiv f'$.*

In 2009, Zhang gave the following noteworthy result

Theorem 3. [19] *Let $f(z) \in \mathfrak{D}$ and $n(\geq 7) \in \mathbb{Z}$. If f^n and $(f^n)'$ share 1 CM, then $f^n \equiv (f^n)'$ and f assumes the form $f(z) = ce^{\left(\frac{z}{n}\right)}$, where $c \neq 0$ is a constant.*

Over time, the initial result evolved, extending from the first derivative to the k^{th} derivative, and the concept of sharing small functions was introduced in the same article, as follows:

Theorem 4. [19] *Let $f(z) \in \mathfrak{D}$, $n,k \in \mathbb{Z}^+$ and $a(z)(\not\equiv 0,\infty)$ be a small function of f. If suppose f^n-a and $(f^n)^{(k)}-a$ share the value 0 CM and $(n-k-1)(n-k-4) > 3k+6$, then $f^n \equiv (f^n)^{(k)}$ and f assumes the form $f(z) = c_1e^{\left(\frac{\lambda z}{n}\right)}$, where c_1 is a non zero constant and $\lambda^k = 1$.*

Theorem 4's outcome was derived under a different prerequisite condition for the sharing of small function a CM and a IM in [20]. Many authors have followed this trend, incorporating various sharing conditions for f^n and $(f^n)^{(k)}$, such as set sharing, two distinct polynomial sharing [10], and so on. For related studies, please refer to [8, 15, 19]. Further contributions to this field include the work of Saha et al. [14], who explored the uniqueness of certain types of shift polynomials sharing a small function. Saha [13] also independently studied the uniqueness of certain types of shift polynomials sharing a small function. Earlier, Majumder [7] had investigated the uniqueness of certain types of shift polynomials sharing a small function, providing a foundation for subsequent research in this area. Lahiri and Majumder, in [6], established the uniqueness theorem for f^n and $(f^n)^{(k)}$ sharing two distinct small functions, by introducing the idea of weighted sharing. The notable result is detailed below:

Theorem 5. [6] *Let the transcendental function $f \in \mathfrak{D}$ such that $N(r,f) = S(r,f)$ and $a_i = a_i(z)(\not\equiv 0,\ \infty)$ be small functions of f, where $i = 1, 2$. Let $n,\ k \in \mathbb{Z}^+$ such that $n \geq k+1$. In addition if $f^n - a_1$ and $(f^n)^{(k)} - a_2$ share $(0,1)$, then $(f^n)^{(k)} \equiv \frac{a_2}{a_1} f^n$. Furthermore, if $a_1 \equiv a_2$, then the conclusion of Theorem 4 holds.*

With the emergence of difference analogue in Nevanlinna theory, uniqueness theorems have also developed correspondingly. Additionally, researchers have devoted considerable attention to investigating the uniqueness properties of differences and difference polynomials of meromorphic functions, yielding several noteworthy findings. Pursuing this line of thought, Majumder-Saha [9] provided the following result:

Theorem 6. [9] *Let the transcendental function $f \in \mathfrak{D}$ be of finite order with finitely many poles. For constant $c(\neq 0) \in \mathbb{C}$, $n,\ k \in \mathbb{N}$, let $f^n(z) - \mathcal{Q}_1$ and $(f^n(z+c))^{(k)} - \mathcal{Q}_2$ share $(0,1)$ and $f(z)$, $f(z+c)$ share 0 CM. If $n \geq k+1$, then $(f^n(z+c))^{(k)} \equiv \frac{\mathcal{Q}_1}{\mathcal{Q}_2} f^n(z)$, where $\mathcal{Q}_1,\ \mathcal{Q}_2$ are polynomials with$\mathcal{Q}_1\mathcal{Q}_2 \not\equiv 0$. Furthermore, if $\mathcal{Q}_1 = \mathcal{Q}_2$, then the conclusion of Theorem 4 holds.*

2. Main Result

An intriguing avenue of investigation is to examine the consequences of Theorem F when we replace $f^n(z)$ with $f^n(z)\Psi_{f(z)}$ and $(f^n(z+c))^{(k)}$ with $\left(f^n(z+c)\Psi_{f(z+c)}\right)^{(k)}$. Drawing inspiration from this, we present the primary result of the paper below:

Throughout this article the term, $\Psi_{f(z)}$, will be defined as

$$\Psi_{f(z)} = a_2 f^2(z) + a_1 f(z) + 1. \tag{2.1}$$

Here, a_2 and a_1 are constants.

Theorem 2.1. *Let the transcendental function $f \in \mathfrak{D}$ be of finite order with finitely many poles. Let $c(\neq 0) \in \mathbb{C}$, $n,\ k \in \mathbb{N}$ and $\Psi_{f(z)}$ be defined as in 2.1. If $f^n(z)\Psi_{f(z)} - \mathcal{P}_1(z)$ and $\left(f^n(z+c)\Psi_{f(z+c)}\right)^{(k)} - \mathcal{P}_2(z)$ share $(0,1)$ such that $f(z)$, $f(z+c)$ share 0 CM with $n \geq k+3$ then $\left(f^n(z+c)\Psi_{f(z+c)}\right)^{(k)} \equiv \frac{\mathcal{P}_1}{\mathcal{P}_2} f^n(z)\Psi_{f(z)}$. Here, $\mathcal{P}_1,\ \mathcal{P}_2$ are polynomials with $\mathcal{P}_1\mathcal{P}_2 \not\equiv 0$. In addition, if $\mathcal{P}_1 = \mathcal{P}_2$, then $f(z) = \mathcal{C}e^{\frac{\varrho z}{n+i}}$ for $i \in \{0,1,2\}$, $C,\ \varrho$ are constant such that $e^{\varrho C} = 1$ and $\varrho^k = 1$.*

Remark 2.1. *When we set $a_2 = a_1 = 0$ in Theorem 2.1, $\Psi_{f(z)}$ reduces to 1, and the conclusion of Theorem 6 remains valid. Consequently, the primary contribution of this paper is an extension of the results previously established in* [9].

Example 2.1. *Consider $f(z) = e^z - 1$, $c = 2\pi i$. Substituting $a_2 = 0,\ a_1 = \frac{1}{2}$ into* (2.1)*, we obtain $\Psi_{f(z)} = \frac{f(z)}{2} + 1$. It is evident that $f(z)$ and $f(z+c)$ share $0\ CM$. For $\mathcal{P}_1 = 1$ and $\mathcal{P}_2 = 8$, both $\left[f(z)\Psi_{f(z)} - \mathcal{P}_1\right]$ and $\left[f(z+c)\Psi_{f(z+c)} - \mathcal{P}_2\right]^{(2)}$ share 0 CM. However, $\left[f(z+c)\Psi_{f(z+c)}\right]^{(2)} \not\equiv \frac{\mathcal{P}_2}{\mathcal{P}_1}\left[f(z)\Psi_{f(z)}\right]$ due to the condition $n \geq k+3$ not being satisfied.*

3. SUPPORTING RESULTS

Lemma 3.1. [2] *Let $f \in \mathfrak{D}$ of finte order and c be non zero complex constant then*

$$m\left(r,\ \frac{f(z+c)}{f(z)}\right)+m\left(r,\ \frac{f(z)}{f(z+c)}\right)=S(r,\ f), \tag{3.2}$$

$$T\left(r, f(z+c)\right)=T(r,\ f)+S(r,\ f). \tag{3.3}$$

Lemma 3.2. [4, Lemma 3.5] *Suppose that F is meromorphic function in a domain D and set $f=\frac{F'}{F}$. Then for $n \geq 1$*

$$\frac{F^{(k)}}{F}=f^n+\frac{n(n-1)}{2}f^{n-2}f'+a_nf^{n-3}f''+b_nf^{n-4}\left(f'\right)^2+P_{n-3}(f)$$

where $a_n=\frac{1}{6}n(n-1)(n-2),\ b_n=\frac{1}{8}n(n-1)(n-2)(n-3)$ and $P_{n-3}(f)$ is a differential polynomial with constant coefficients, which vanishes identically for $n \leq 3$ and has degree $n-3$ when $n>3$.

Lemma 3.3. [5] *Let $f \in \mathfrak{D}$ of finte order and c be non zero complex constant. Let $P(f,z)$ be a polynomial in $f(z+c)$ and its derivatives and $Q(z,f)$ be a polynomial in $f(z),\ f(z+c)$ and its derivatives with meromorphic coefficients $a_\lambda,\ \lambda \in \mathbb{Z}$ such that $m(r,\ a_\lambda)=S(r,f)$. If $f^nP(z,f)=Q(z,f)$ and the total degree of $Q(z,f)$ is n then*

$$m(r,P(z,f))=S(r,f).$$

4. PROOF OF THEOREM

Let

$$\mathcal{F}=f^n(z)\Psi_{f(z)}\ \text{ and }\ \mathcal{G}=\left[f^n(z+c)\Psi_{f(z+c)}\right]^{(k)}. \tag{4.4}$$

Set

$$\mathcal{F}_1(z)=\frac{\mathcal{F}}{\mathcal{P}_1(z)}\ \text{ and }\ \mathcal{G}_1(z)=\frac{\mathcal{G}}{\mathcal{P}_2(z)} \tag{4.5}$$

Disregarding the zeros of $\mathcal{P}_i(z),\ \ i=1,2$, it is evident that $\mathcal{F}_1(z)$ and $\mathcal{G}_1(z)$ share $(1,1)$. As a result, $\overline{N}\left(r, \frac{1}{\mathcal{F}_1-1}\right)=\overline{N}\left(r, \frac{1}{\mathcal{G}_1-1}\right)+S(r,f)$.
By Lemma 3.1, we conclude that $m\left(r,\ \frac{\mathcal{G}}{\mathcal{F}}\right)=S(r,f)$.
Define

$$\phi=\frac{\mathcal{F}_1'(\mathcal{F}_1-\mathcal{G}_1)}{\mathcal{F}_1(\mathcal{F}_1-1)}. \tag{4.6}$$

Case 1. Let's start by considering that ϕ is not identically zero. Clearly, $m(r,\phi)$ equals $S(r,f)$. Let's take z_0 as a zero of $f(z)$ with a multiplicity of p (greater than or equal to 1) and as a zero of $\Psi_{f(z)}$ with a multiplicity q (greater than or equal to 1), excluding the zeros of $\mathcal{P}_i,\ i=1,2$. Since $f(z)$ and $f(z+c)$ share 0 CM, it is observed that z_0 is also a zero of $f(z+c)$ with the same multiplicity p, and it is a zero of $\Psi_{f(z+c)}$ with the same multiplicity q.
Based on equation (4.5), z_0 will serve as a zero for both $\mathcal{F}_1$ and $\mathcal{G}_1$, possessing multiplicities of $np+q$ and $mp+q-k$ respectively. Given this context, equation (4.6) can be reformulated as follows:

$$\phi(z)=O\left((z-z_0)^{np+q-k-1}\right). \tag{4.7}$$

Given that $n \geq k+2$, it is observable that $\phi(z)$ is holomorphic at z_0. Suppose z_1 is a zero of $\mathcal{F}_1-1$ with multiplicity $q_1(\geq 2)$, excluding the zeros of $\Psi_{f(z)}$ and $\mathcal{P}_i(z),\ i=1,2$. Since $\mathcal{F}_1$ and $\mathcal{G}_1$ share $(1,1)$, it is evident that z_1 is also a zero of $\mathcal{G}_1-1$ with a multiplicity

of $r_1(\geq 2)$.
In the neighbourhood of z_1 the taylor series expansion of functions will be as follows:

$$\mathcal{F}_1(z) - 1 = a_{q_1}(z-z_1)^{q_1} + a_{q_1+1}(z-z_1)^{q_1+1} + \cdots, a_{q_1} \neq 0,$$

$$\mathcal{G}_1(z) - 1 = b_{r_1}(z-z_1)^{r_1} + b_{r_1+1}(z-z_1)^{r_1+1} + \cdots, b_{r_1} \neq 0,$$

$$\mathcal{F}_1(z) - \mathcal{G}_1(z) = \begin{cases} a_{q_1}(z-z_1)^{q_1} + a_{q_1+1}(z-z_1)^{q_1+1} + \cdots, & \text{if } q_1 < r_1, \\ -(b_{r_1}(z-z_1)^{r_1} + b_{r_1+1}(z-z_1)^{r_1+1} + \cdots) & \text{if } q_1 > r_1, \\ (a_{r_1} - b_{r_1})(z-z_1)^{r_1} + (a_{r_1+1} - b_{r_1+1})(z-z_1)^{r_1+1} + \cdots, & \text{if } q_1 = r_1, \end{cases}$$

$$\implies \mathcal{F}'(z) = q_1 a_{q_1}(z-z_1)^{q_1-1} + (q_1+1)a_{q_1+1}(z-z_1)^{q_1} + \cdots$$

Let $t_1 \geq \max\{q_1,\ r_1\} \geq 2$ with this regard (4.6) can be rewritten as

$$\phi(z) = O\left((z-z_1)^{t_1-1}\right) \tag{4.8}$$

It is evident that $\phi(z)$ is holomorphic at z_1. The zeros of $\mathcal{P}_i$, where $i = 1, 2$, and the poles of $f(z)$ collectively contribute to the poles of $\phi(z)$. Consequently, $\phi(z)$ possesses a finite number of poles, implying that $N(r,\phi) = O(\log r)$, and thus $T(r,\phi) = S(r,f)$. From (4.8), we see that

$$\overline{N}_{(2}\left(r,\ \frac{1}{\mathcal{F}_1 - 1}\right) \leq N\left(r,\ \frac{1}{\phi}\right) \leq T(r,\phi) + S(r,f),$$

$$\implies \overline{N}_{(2}\left(r,\ \frac{1}{\mathcal{F}_1 - 1}\right) = S(r,f)$$

Once more, since $\mathcal{F}_1$ and $\mathcal{G}_1$ share $(1,1)$ except for the zeros of $\mathcal{P}_i(z)$, where $i = 1, 2$, we observe that $\overline{N}_{(2}\left(r,\ \frac{1}{\mathcal{G}_1 - 1}\right) = S(r,f)$.
Rearranging the terms in (4.6), we get

$$\frac{1}{\mathcal{F}_1} = \frac{\mathcal{F}_1'}{\phi\mathcal{F}_1(\mathcal{F}_1 - 1)}\left(1 - \frac{\mathcal{G}_1}{\mathcal{F}_1}\right) \tag{4.9}$$

From (4.5), we get $\frac{\mathcal{G}_1}{\mathcal{F}_1} = \frac{\mathcal{P}_1\mathcal{G}}{\mathcal{P}_2\mathcal{F}}$. Hence

$$m\left(r, \frac{1}{\mathcal{F}_1}\right) = S(r,f) \ \text{ and } \ m\left(r, \frac{1}{f}\right) = S(r,f). \tag{4.10}$$

Case 1.1. Suppose $n > k+3$, with reference to (4.7), we see that

$$N\left(r, \frac{1}{f}\right) \leq N\left(r, \frac{1}{\phi}\right) \leq T(r, \frac{1}{\phi}) \leq T(r,\phi) + S(r,f) \tag{4.11}$$

Combining (4.10) and (4.11), we get

$$\implies T(r,f) = S(r,f) \tag{4.12}$$

Which is contradiction
Case 1.2. Suppose $n = k+3$ with reference to (4.7), we see that

$$\overline{N}_{(2}\left(r,\ \frac{1}{f}\right) \leq N\left(r,\ \frac{1}{\phi}\right) \leq T(r,\phi) + S(r,f),$$

but $T(r,f) = N\left(r, \frac{1}{f}\right) + m\left(r, \frac{1}{f}\right)$, hence

$$T(r,f) = N_{1)}\left(r, \frac{1}{f}\right) + S(r,f) \tag{4.13}$$

It is evident that,

$$\overline{N}_{(2}\left(r, \frac{1}{\mathcal{F}-\mathcal{P}_1}\right) = S(r,f) \text{ and } \overline{N}_{(2}\left(r, \frac{1}{\mathcal{G}-\mathcal{P}_2}\right) = S(r,f) \tag{4.14}$$

by the definition of $\mathcal{F}$ and $\mathcal{G}$.
Since $\mathcal{F}-\mathcal{P}_1$ and $\mathcal{G}-\mathcal{P}_2$ share $(0,1)$, there exists a meromorphic function, denoted $\mathfrak{Z}$, of finite order such that

$$\frac{\mathcal{G}-\mathcal{P}_2}{\mathcal{F}-\mathcal{P}_1} = \mathfrak{Z} \text{ or } \mathcal{G}-\mathcal{P}_2 = \mathfrak{Z}\left(\mathcal{F}-\mathcal{P}_1\right) \tag{4.15}$$

Case 1.2.1. Now let's examine the scenario where $\mathfrak{Z}$ is not constant. Assume z_2 is a zero of $\mathfrak{Z}$. Given that $\mathcal{F}-\mathcal{P}_1$ and $\mathcal{G}-\mathcal{P}_2$ share $(0,1)$, it's evident that z_2 is a zero of $\mathcal{F}-\mathcal{P}_1$ with a multiplicity, let's say r_2, and z_2 is a zero of $\mathcal{G}-\mathcal{P}_2$ with a multiplicity, let's call it q_2, such that $r_2 < q_2$. If $r_2 > q_2$, then z_2 becomes a pole of $\mathfrak{Z}$. Given the finite poles of $\mathcal{F}$ and $\mathcal{G}$, it follows that $N(r,\mathcal{F}) = S(r,f)$ and $N(r,\mathcal{G}) = S(r,f)$.
Therefore using (4.14) and (4.15), we can write

$$\overline{N}\left(r, \frac{1}{\mathfrak{Z}}\right) \le \overline{N}_{(2}\left(r, \frac{1}{\mathcal{G}-\mathcal{P}_2}\right) + S(r,f), \quad \overline{N}\left(r,\mathfrak{Z}\right) \le \overline{N}_{(2}\left(r, \frac{1}{\mathcal{F}-\mathcal{P}_1}\right) + S(r,f).$$

Differentiating (4.15), we get

$$\mathcal{G}' - \mathcal{P}_2' = \mathfrak{Z}'\left(\mathcal{F}-\mathcal{P}_1\right) + \mathfrak{Z}\left(\mathcal{F}'-\mathcal{P}_1'\right) \tag{4.16}$$

In (4.16), replace the term $\mathcal{F}-\mathcal{P}_1$ and $\mathcal{F}$ from (4.15) and then rearranging, we arrive at

$$\begin{aligned}\mathcal{G}'\mathcal{F} - \mathcal{G}\mathcal{F}' - \frac{\mathfrak{Z}'}{\mathfrak{Z}}\mathcal{G}\mathcal{F} =& \mathcal{P}_1\mathcal{G} + \left(\mathcal{P}_2' - \frac{\mathfrak{Z}'}{\mathfrak{Z}}\mathcal{P}_2\right)\mathcal{F} - \left(\frac{\mathfrak{Z}'}{\mathfrak{Z}}\mathcal{P}_1 + \mathcal{P}_1'\right)\mathcal{G} - \mathcal{P}_2\mathcal{F}' \\ &+ \frac{\mathfrak{Z}'}{\mathfrak{Z}}\mathcal{P}_1\mathcal{P}_2 - \mathcal{P}_1\mathcal{P}_2' + \mathcal{P}_1'\mathcal{P}_2\end{aligned} \tag{4.17}$$

Let $\mathcal{B} = \frac{\mathfrak{Z}'}{\mathfrak{Z}}$, leading to $T(r,\mathcal{B}) = S(r,f)$. Since $f(z)$ has a finite number of poles and shares 0 CM with $f(z+c)$. Hence

$$f(z) = f(z+c)\psi(z)e^{\gamma(z)} \text{ (or) } \frac{f(z)}{f(z+c)} = \psi(z)e^{\gamma(z)} \tag{4.18}$$

where $\gamma(z)$ and $\psi(z)$ is a polynoamial and rational function respectively.
Differentiating (4.18), we get

$$f'(z) = f'(z+c)\psi(z)e^{\gamma(z)} + f(z+c)\psi'(z)e^{\gamma(z)} + f(z+c)\psi(z)e^{\gamma(z)}\gamma'(z). \tag{4.19}$$

Dividing (4.19) by (4.18),

$$\frac{f'(z)}{f(z)} = \frac{f'(z+c)}{f(z+c)} + \frac{\psi'(z)}{\psi(z)} + \gamma'(z) = \frac{f'(z+c)}{f(z+c)} + \mathcal{N}(z)$$

where $\mathcal{N}(z) = \frac{\psi'(z)}{\psi(z)} + \gamma'(z)$. Using the Lemma 3.1, from (4.19) we get

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = S(r,f) \text{ and } T(r,\psi e^{\gamma}) = S(r,f).$$

From (4.4), we have

$$\mathcal{G}(z) = \left\{a_2 f^{n+2}(z+c) + a_1 f^{n+1}(z+c) + f^n(z+c)\right\}^{(k)} \tag{4.20}$$

For $k = 1$, we get

$$\mathcal{G}(z) = \left\{a_2(n+2)f^{n+1}(z+c) + a_1(n+1)f^n(z+c) + nf^{n-1}(z+c)\right\} f'(z+c).$$

For $k=2$, we get

$$\begin{aligned}\mathcal{G}(z) =&\{a_2[(n+2)(n+1)f^n(z+c)(f'(z+c))^2+(n+2)f^{n+1}(z+c)f''(z+c)]\\ &+a_1[(n+1)f^n(z+c)f''(z+c)+n(n+1)f^{n-1}(z+c)(f''(z+c))^2]\\ &+a_0[n(n-1)f^{n-2}(z+c)(f'(z+c))^2+nf^{n-1}(z+c)f''(z+c)]\}.\end{aligned}$$

In general differentiating $k-$ times, we get

$$\begin{aligned}\mathcal{G}(z) =&\sum_{\lambda^2} a_{\lambda^2}\left(f(z+c)\right)^{\vartheta_0^{\lambda^2}}\left(f'(z+c)\right)^{\vartheta_1^{\lambda^2}}\cdots\left(f^{(k)}(z+c)\right)^{\vartheta_k^{\lambda^2}}\\ &+\sum_{\lambda^1} a_{\lambda^1}\left(f(z+c)\right)^{\vartheta_0^{\lambda^1}}\left(f'(z+c)\right)^{\vartheta_1^{\lambda^1}}\cdots\left(f^{(k)}(z+c)\right)^{\vartheta_k^{\lambda^1}}\\ &+\sum_{\lambda^0}\left(f(z+c)\right)^{\vartheta_0^{\lambda^0}}\left(f'(z+c)\right)^{\vartheta_1^{\lambda^0}}\cdots\left(f^{(k)}(z+c)\right)^{\vartheta_k^{\lambda^0}}\end{aligned}\tag{4.21}$$

where $\vartheta_0^{\lambda^i},\ \vartheta_1^{\lambda^i},\cdots\vartheta_K^{\lambda^i}\in\mathbb{Z}^+$ such that $\sum\limits_{j=0}^{k}\vartheta_j^{\lambda^i}=n+i$ and $n+i-k\leq\vartheta_0^{\lambda^i}\leq n+i-1$ for $i=1,2$ and $a_{\lambda^1},\ a_{\lambda^2}$ are constants.
Differentiating the above equation, we get

$$\begin{aligned}\mathcal{G}'(z) =&\sum_{\lambda^2} b_{\lambda^2}\left(f(z+c)\right)^{d_0^{\lambda^2}}\left(f'(z+c)\right)^{d_1^{\lambda^2}}\cdots\left(f^{(k+1)}(z+c)\right)^{d_{k+1}^{\lambda^2}}\\ &+\sum_{\lambda^1} b_{\lambda^1}\left(f(z+c)\right)^{d_0^{\lambda^1}}\left(f'(z+c)\right)^{d_1^{\lambda^1}}\cdots\left(f^{(k+1)}(z+c)\right)^{d_{k+1}^{\lambda^1}}\\ &+\sum_{\lambda^0}\left(f(z+c)\right)^{d_0^{\lambda^0}}\left(f'(z+c)\right)^{d_1^{\lambda^0}}\cdots\left(f^{(k+1)}(z+c)\right)^{d_{k+1}^{\lambda^0}}\end{aligned}\tag{4.22}$$

where $d_0^{\lambda^i},\ d_1^{\lambda^i},\cdots d_K^{\lambda^i}\in\mathbb{Z}^+$ such that $\sum\limits_{j=0}^{k+1}d_j^{\lambda^i}=n+i$ and $n+i-k\leq d_0^{\lambda^i}\leq n+i-1$ for $i=1,2$ and $b_{\lambda^1},\ b_{\lambda^2}$ are constants.
From (4.4), we have

$$\mathcal{F}(z)=f^{n+2}(z)\left[a_2+\frac{a_1}{f(z)}+\frac{1}{f^2(z)}\right]\tag{4.23}$$

Differentiating (4.23), we get

$$\mathcal{F}'(z)=f^{n+2}f'\left[\frac{a_2(n+2)}{f(z)}+\frac{(n+1)a_1}{f^2(z)}+\frac{n}{f^3(z)}\right]\tag{4.24}$$

Substituting (4.21)-(4.24) in (4.17), we see that
(4.25)

$$f^{n+2}(z)\left\{(\mathcal{G}'-\mathcal{B}\mathcal{G})\left[a_2+\frac{a_1}{f(z)}+\frac{1}{f^2(z)}\right]-\mathcal{G}\frac{f'(z)}{f(z)}\left[a_2(n+2)+\frac{(n+1)a_1}{f(z)}+\frac{n}{f^2(z)}\right]\right\}=\mathcal{Q}(z)$$

where $\mathcal{Q}(z)$ is a differential polynomial in $f(z)$ and $f(z+c)$ of degree n.
Let

$$\mathfrak{R}_1(z) = a_2 + \frac{a_1}{f(z)} + \frac{1}{f^2(z)}\ ; \ \ \mathfrak{R}_2(z) = a_2(n+2) + \frac{(n+1)a_1}{f(z)} + \frac{n}{f^2(z)},$$

$$\mathcal{G}'(z)\mathfrak{R}_1(z) - \mathcal{B}\mathcal{G}(z)\mathfrak{R}_1(z) - \mathcal{G}(z)\frac{f'(z)}{f(z)}\mathfrak{R}_2(z) = \mathfrak{P}(z) \tag{4.26}$$

Using (4.26) in (4.25), we get

$$f^{n+2}(z)\mathfrak{P}(z) = \mathcal{Q}(z). \tag{4.27}$$

From (4.20), $\frac{f'(z)}{f(z)}$ can be replaced by $\frac{f'(z+c)}{f(z+c)} + \frac{\psi'(z)}{\psi(z)} + \gamma'(z)$. Hence from (4.26),

$$\begin{aligned}\mathfrak{P}(z) =&\mathfrak{R}_1(z)\Big\{\sum_{\lambda^2} b_{\lambda^2}\left(f(z+c)\right)^{d_0^{\lambda^2}}\left(f'(z+c)\right)^{d_1^{\lambda^2}}\cdots\left(f^{(k+1)}(z+c)\right)^{d_k^{\lambda^2}}\\ &+\sum_{\lambda^1} b_{\lambda^1}\left(f(z+c)\right)^{d_0^{\lambda^1}}\left(f'(z+c)\right)^{d_1^{\lambda^1}}\cdots\left(f^{(k+1)}(z+c)\right)^{d_k^{\lambda^1}}\\ &+\sum_{\lambda^0}\left(f(z+c)\right)^{d_0^{\lambda^0}}\left(f'(z+c)\right)^{d_1^{\lambda^0}}\cdots\left(f^{(k+1)}(z+c)\right)^{d_k^{\lambda^0}}\Big\}\\ &-f'(z+c)\mathfrak{R}_2(z)\Big\{\sum_{\lambda^2} a_{\lambda^2}\left(f(z+c)\right)^{\vartheta_0^{\lambda^2}-1}\left(f'(z+c)\right)^{\vartheta_1^{\lambda^2}}\cdots\left(f^{(k)}(z+c)\right)^{\vartheta_k^{\lambda^2}}\\ &+\sum_{\lambda^1} a_{\lambda^1}\left(f(z+c)\right)^{\vartheta_0^{\lambda^1}-1}\left(f'(z+c)\right)^{\vartheta_1^{\lambda^1}}\cdots\left(f^{(k)}(z+c)\right)^{\vartheta_k^{\lambda^1}}\\ &+\sum_{\lambda^0}\left(f(z+c)\right)^{\vartheta_0^{\lambda^0}-1}\left(f'(z+c)\right)^{\vartheta_1^{\lambda^0}}\cdots\left(f^{(k)}(z+c)\right)^{\vartheta_k^{\lambda^0}}\Big\} - \mathcal{G}(z)\left[\frac{\psi'(z)}{\psi(z)}+\gamma'(z)\right]\mathfrak{R}_2(z).\end{aligned}$$

The expression $\mathcal{G}(z)\left[\frac{\psi'(z)}{\psi(z)}+\gamma'(z)\right]\mathfrak{R}_2(z)$ in the given equation does not include the highest power of $f'(z+c)$ and can thus be disregarded. In general

$$\mathfrak{P}(z) = \mathcal{H}\left[f'(z+c)\right]^{k+1} + \mathcal{I}_*(f) \tag{4.28}$$

Here, $\mathcal{H}$ represents a suitable constant, and $\mathcal{I}_*(f)$ is a polynomial, specifically in the form

$$\begin{aligned}\mathcal{I}_*(f) =&S(\mathcal{B},\mathcal{B}',\psi,\psi',\gamma')\Big\{\sum_{\lambda^2}\left(f(z+c)\right)^{u_0^{\lambda^2}}\left(f'(z+c)\right)^{u_1^{\lambda^2}}\cdots\left(f^{(k+1)}(z+c)\right)^{u_{k+1}^{\lambda^2}}\\ &+\sum_{\lambda^1}\left(f(z+c)\right)^{u_0^{\lambda^1}}\left(f'(z+c)\right)^{u_1^{\lambda^1}}\cdots\left(f^{(k+1)}(z+c)\right)^{u_{k+1}^{\lambda^1}}\\ &+\sum_{\lambda^0}\left(f(z+c)\right)^{u_0^{\lambda^0}}\left(f'(z+c)\right)^{u_1^{\lambda^0}}\cdots\left(f^{(k+1)}(z+c)\right)^{u_{k+1}^{\lambda^0}}\Big\}\end{aligned}$$

Here, $u_0^{\lambda^i},\ u_1^{\lambda^i},\ldots,u_{K+1}^{\lambda^i}\in\mathbb{Z}^+$ such that $\sum_{j=0}^{k+1} u_j^{\lambda^i} = n+2i$ and $n+2i-k \le u_0^{\lambda^i} \le n+2i-1$ for $i=1,2$. Additionally, $S(\mathcal{B},\mathcal{B}',\psi,\psi',\gamma')$ represents a polynomial in $\mathcal{B},\mathcal{B}',\psi,\psi',\gamma'$ with constant coefficients.

With respect to (4.27), we consider two cases:
Case 1.2.1.1. Suppose $\mathfrak{P}(z) \not\equiv 0$. Using Lemma 3.1 we see that

$$m(r, \mathfrak{P}) = S(r, f) \text{ and } T(r, \mathfrak{P}') = S(r, f) \tag{4.29}$$

Differentiating (4.28), we get

$$\mathfrak{P}'(z) = \mathcal{H}(k+1)\left[f'(z+c)\right]^k f''(z+c) + LS(z)\left[f'(z+c)\right]^{k+1} + S_1(z) \tag{4.30}$$

where L is a suitable constant $S(z) = S(\mathcal{B},\ \mathcal{B}',\ \psi,\ \psi',\ \gamma')$ and $S_1(z)$ is a polynomial of the form

$$\begin{aligned} S_1(z) =& S(z)\Big\{ \sum_{\lambda^2} (f(z+c))^{v_0^{\lambda^2}} (f'(z+c))^{v_1^{\lambda^2}} \cdots \left(f^{(k+1)}(z+c)\right)^{v_{k+1}^{\lambda^2}} \\ &+ \sum_{\lambda^1} (f(z+c))^{v_0^{\lambda^1}} (f'(z+c))^{v_1^{\lambda^1}} \cdots \left(f^{(k+1)}(z+c)\right)^{v_{k+1}^{\lambda^1}} \\ &+ \sum_{\lambda^0} (f(z+c))^{v_0^{\lambda^0}} (f'(z+c))^{v_1^{\lambda^0}} \cdots \left(f^{(k+1)}(z+c)\right)^{v_{k+1}^{\lambda^0}} \Big\} \end{aligned}$$

where $v_0^{\lambda^i},\ v_1^{\lambda^i}, \ldots, v_{K+1}^{\lambda^i} \in \mathbb{Z}^+$ such that $\sum_{j=0}^{k+1} v_j^{\lambda^i} = n+2i$ and $n+2i-k \leq v_0^{\lambda^i} \leq n+2i-1$ for $i = 1, 2$. Asume z_3 to be a simple zero of $f(z+c)$ except for the zeros of $\mathcal{G}$ and $\mathcal{G}'$. So (4.28) and (4.30) can be written as

$$\mathfrak{P}(z_3) = \mathcal{H}\left[f'(z_3+c)\right]^{k+1} \tag{4.31}$$

$$\mathfrak{P}'(z_3) = \mathcal{H}(k+1)\left[f'(z_3+c)\right]^k f''(z_3+c) + LS(z_3)\left[f'(z_3+c)\right]^{k+1} \tag{4.32}$$

Uing (4.31) in (4.32) and then rearranging, we get

$$\mathfrak{P}(z_3) f''(z_3+c) - \frac{\mathfrak{P}'(z_3) f'(z_3)}{k+1} + \frac{LS(z_3) f'(z_3+c)}{\mathcal{H}(k+1)} = 0 \tag{4.33}$$

Let $\mathcal{K}_1 = \frac{1}{k+1}$ and $\mathcal{K}_2 = \frac{LS(z_3)}{\mathcal{H}(k+1)}$. So (4.33) becomes

$$\mathfrak{P}(z_3) f''(z_3+c) - \mathcal{K}_1 \mathfrak{P}'(z_3) f'(z_3+c) + \mathcal{K}_2 f'(z_3+c) = 0.$$

Clearly z_3 is a zero of $\mathfrak{P}(z) f''(z_3+c) - \mathcal{K}_1\mathfrak{P}'(z_3) f'(z_3+c) + \mathcal{K}_2 f'(z_3+c)$ and consequently $T(r, \mathcal{K}_1) = S(r, f)$ and $T(r, \mathcal{K}_2) = S(r, f)$.
Let us define

$$\Phi_1(z) = \frac{\mathfrak{P}(z) f''(z+c) - \mathcal{K}_1 \mathfrak{P}'(z) f'(z+c) + \mathcal{K}_2 f'(z+c)}{f(z+c)} \tag{4.34}$$

Let

$$\mathcal{V}_1(z) = \frac{\Phi_1(z)}{\mathfrak{P}(z)} \text{ and } \mathcal{V}_2(z) = \frac{\mathcal{K}_1(z)\mathfrak{P}'(z)}{\mathfrak{P}(z)} - \mathcal{K}_2(z) \tag{4.35}$$

Using (4.35), (4.34) can be written as

$$f''(z+c) = \mathcal{V}_1(z) f(z+c) + \mathcal{V}_2(z) f'(z+c). \tag{4.36}$$

Clearly $T(r, \mathcal{V}_1) = S(r, f)$ and $T(r, \mathcal{V}_2) = S(r, f)$.
Suppose if $\Phi_1(z) \equiv 0$ then $\mathcal{V}_1(z) = 0$, the detailed analysis of this case is on the same line

of the equation (3.24) in [9].
Let us suppose if $\Phi_1(z) \not\equiv 0$, then from (4.35) we have

$$\mathfrak{P}'(z) = \left[\frac{\mathcal{V}_2(z)}{\mathcal{K}_1(z)} + \frac{\mathcal{K}_2(z)}{\mathcal{K}_1(z)}\right]\mathfrak{P}(z). \tag{4.37}$$

Substituting (4.28) in (4.37), we get

$$\mathfrak{P}'(z) = \left[\frac{\mathcal{V}_2(z)}{\mathcal{K}_1(z)} + \frac{\mathcal{K}_2(z)}{\mathcal{K}_1(z)}\right]\mathcal{H}\left[f'(z+c)\right]^{k+1} + \left[\frac{\mathcal{V}_2(z)}{\mathcal{K}_1(z)} + \frac{\mathcal{K}_2(z)}{\mathcal{K}_1(z)}\right]\mathcal{I}_*(f). \tag{4.38}$$

Substituting (4.36) in (4.30), we get

$$\begin{aligned}\mathfrak{P}'(z) = \mathcal{H}(k+1)\mathcal{V}_1(z)\left[f'(z+c)\right]^k f(z+c) + \left[\mathcal{H}(k+1)\mathcal{V}_2(z) + LS(z)\right]\left[f'(z+c)\right]^{k+1} \\ + S_1(z)\end{aligned} \tag{4.39}$$

Comparing equation (4.38) and (4.39) we see that

$$\begin{aligned}&\left[\mathcal{H}\left(\frac{\mathcal{V}_2(z)}{\mathcal{K}_1(z)} + \frac{\mathcal{K}_2(z)}{\mathcal{K}_1(z)}\right) - \mathcal{H}(k+1)\mathcal{V}_2(z) - LS(z)\right]\left[f'(z+c)\right]^{k+1} \\ &- \mathcal{H}(k+1)\mathcal{V}_1(z)\left[f'(z+c)\right]^k f(z+c) + \left[\frac{\mathcal{V}_2(z)}{\mathcal{K}_1(z)} + \frac{\mathcal{K}_2(z)}{\mathcal{K}_1(z)}\right]\mathcal{I}_*(f) - S_1(z) \equiv 0.\end{aligned}$$

Since $\mathcal{V}_1(z) \not\equiv 0$, from (4.39) we have

$$N_{1)}\left(r, \frac{1}{f}\right) = S(r, f). \tag{4.40}$$

Using equations (4.13) and (4.40), we see that $T(r, f) = S(r, f)$, Which is a contradiction.
Case 1.2.1.2. Suppose $\mathfrak{P}(z) \equiv 0$. From (4.27), we see that $\mathcal{Q}(z) \equiv 0$ hence (4.17) becomes

$$\mathcal{G}'\mathcal{F} - \mathcal{G}\mathcal{F}' - \frac{\mathfrak{Z}'}{\mathfrak{Z}}\mathcal{G}\mathcal{F} \equiv 0 \quad \textbf{or} \quad \frac{\mathcal{G}'}{\mathcal{G}} = \frac{\mathfrak{Z}'}{\mathfrak{Z}} + \frac{\mathcal{F}'}{\mathcal{F}} \tag{4.41}$$

Upon integration of the above equation, we obtain $\mathcal{G} = l\mathfrak{Z}\mathcal{F}$ where l is a nonzero constant. Given that $n = k+3$ and $\overline{N}(r, \mathfrak{Z}) = S(r, f)$, from (4.15) it follows that $\overline{N}\left(r, \frac{1}{f}\right) = S(r, f)$. Consequently, from (4.15), $T(r, f) = S(r, f)$, which leads to a contradiction.
Case 1.2.2. Let us consider the case when $\mathfrak{Z}$ is a constant say $\mathcal{A}$ such that $\mathcal{A} \neq 0$. From (4.15), we can write

$$\mathcal{G} - \mathcal{P}_2 = \mathcal{A}(\mathcal{F} - \mathcal{P}_1) \quad \text{(or)} \quad \mathcal{G} - \mathcal{A}\mathcal{F} = \mathcal{P}_2 - \mathcal{A}\mathcal{P}_1 \tag{4.42}$$

We have $n = k+3$, it follows that $\overline{N}\left(r, \frac{1}{f}\right) = S(r, f)$ and consequently from (4.13), $T(r, f) + S(r, f)$. Which leads to contradiction.
Case 2. Suppose $\phi \equiv 0$. From (4.6), we get $\mathcal{F}_1 \equiv \mathcal{G}_1$ i.e.,

$$\left[f^n(z+c)\Psi_{f(z+c)}\right]^{(k)} \equiv \frac{\mathcal{P}_2}{\mathcal{P}_2}\left(f^n(z)\Psi_{f(z)}\right) \tag{4.43}$$

Furthermore if $\mathcal{P}_1 \equiv \mathcal{P}_2$, then

$$\left[f^n(z+c)\Psi_{f(z+c)}\right]^{(k)} \equiv \left(f^n(z)\Psi_{f(z)}\right) \tag{4.44}$$

Let's assume z_4 to be the zero of $f(z)$ with multiplicity say r^*. As $f(z)$ and $f(z+c)$ share 0 CM. z_4 will be a zero of $f(z+c)$ with multiplicity t^*, z_4 is a zero of $\left(f^n(z)\Psi_{f(z)}\right)$ with multiplicity $(n+2)r^*$ and z_4 is a zero of $\left[f^n(z+c)\Psi_{f(z+c)}\right]^{(k)}$ with multiplicity $(n+2)r^* - k$. This will be a contradiction in the backdrop of (4.44). As a result we have $f(z) \neq 0$, $f(z+c) \neq 0$. Let

$$\mathcal{G}_1(z) = f^n(z+c)\Psi_{f(z+c)} \tag{4.45}$$

Clearly from (4.45), $(\mathcal{G}_1(z))^{(k)} \neq 0$ [as $f(z) \neq 0$ and $f(z+c) \neq 0$]. Since $f(z)$ is a transcendental meromorphic function with finitely many poles and $f(z) \neq 0$, $f(z)$ must take the form

$$f(z) = \frac{1}{\mathcal{L}_1(z)} e^{\mathcal{L}_2(z)} \tag{4.46}$$

where $\mathcal{L}_1(z)$ is a non-zero polynomial and $\mathcal{L}_2(z)$ is a non constant polynomial. Therefore

$$\mathcal{G}_1(z) = \frac{a_i e^{\mathcal{L}_4(z)}}{\mathcal{L}_3(z)}, \text{ where } \mathcal{L}_3(z) = \mathcal{L}_1^{n+i}(z+c),\ \mathcal{L}_4 = (n+i)\mathcal{L}_2(z+c) \tag{4.47}$$

Define

$$\varsigma(z) = \frac{\mathcal{G}_1'(z)}{\mathcal{G}_1(z)} = \mathcal{L}_4'(z) - \frac{\mathcal{L}_3'(z)}{\mathcal{L}_3(z)}. \tag{4.48}$$

Using (4.49) in Lemma 3.2, we get

$$\frac{\mathcal{G}_1^{(k)}(z)}{\mathcal{G}_1(z)} = \varsigma^k(z) + Q_{k-1}(\varsigma) \tag{4.49}$$

where $Q_{k-1}(z)(\varsigma)$ is a polynomial of degree $k-1$ in ς and its derivative.
If $\mathcal{L}_4'$ is not a constant, we see that

$$\frac{\mathcal{G}_1^{(k)}(z)}{\mathcal{G}_1(z)} \sim \varsigma^k \sim (\mathcal{L}_4'(z))^k \to \infty \ \ as \ \ z \to \infty,$$

we know that every non-constant rational function assumes every value in the closed complex plane. Consequently $(\mathcal{G}_1(z))^{(k)} = 0$ somewhere in the open complex plane. Therefore we arrive at a contradiction.
Next we suppose that $\mathcal{L}_4'$ is a constant. Let $\mathcal{L}_4' = \varrho \neq 0$. If $\varsigma(z)$ is non-constant, then we see that $\varsigma(z) = \varrho$, $\varsigma' = \varsigma'' = \cdots = 0$ at ∞. Also by Lemma 3.2, we observe that $\frac{\mathcal{G}_1^{(k)}(z)}{\mathcal{G}_1(z)} = \varrho^k,\ z \to \infty$. Again $\frac{\mathcal{G}_1^{(k)}(z)}{\mathcal{G}_1(z)}$ must have a zero in the open complex plane. Consequently ς is aconstant. Therefore $\mathcal{L}_4(z) = \varrho(z) = \varsigma(z)$. From (4.49) we get

$$\mathcal{G}_1(z) = e^{\varrho z + d}, \tag{4.50}$$

where d is a constant and consequntly from (4.45) we get $f(z) = \mathcal{C}e^{\left(\frac{\varrho z}{n+i}\right)}$ where $\mathcal{C}(\neq 0) = \frac{c_*}{a_i}$, $i = 0,1,2$ and $a_0 = 1$ is a constant such that $e^{\varrho \mathcal{C}} = 1$, c_* is an integration constant and $\varrho^k = 1$.

5. Conclusion

This theorem contributes to the ongoing research on meromorphic functions by establishing a significant uniqueness theorem concerning shift polynomials and derivative-sharing polynomials. The provided illustrative examples highlight the importance and relevance of the conditions imposed in this context, further supporting the extension of existing literature.

Open questions:

(1) What modifications occur to Theorem 2.1 when examined through the lens of weakly weighted sharing and truncated sharing, which are less restrictive concepts than weighted sharing?
(2) How do the result change when $\Psi_{f(z)}$ is considered to be a non-zero polynomial of higher degree?

Applications The result obtained can be used in numerical methods for approximating solutions to complex equations involving meromorphic functions.

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