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Diffeomorphic embedding of Higher-dimensional Hilbert Manifolds into Hilbert spaces

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ABSTRACT. This manuscript investigates the relationship between higher-dimensional Hilbert Manifolds and Hilbert spaces using the diffeomorphic embedding theorem. The existence of diffeomorphisms between Hilbert spaces will be demonstrated, and we will provide exact constructions that demonstrate their mathematical flexibility in several scenarios. The more general case of compact complex Manifolds equipped with Hermitian metrics is investigated. Under specific conditions, the existence of diffeomorphic embeddings into a separable infinite-dimensional Hilbert space that preserve both the complex structure and the Hermitian geometry is demonstrated. These results pave the way for studying complex geometry within the framework of infinite-dimensional Hilbert spaces.

1. INTRODUCTION

A Hilbert space is a linear space with an inner product that supports length and angle concepts, and it is complete in the sense that any convergent sequence has a limit within the space. A Hilbert Manifold is a smooth Manifold containing a Hilbert space. It is a smooth Manifold with each tangent space being a Hilbert space and smooth charts from Hilbert spaces providing local coordinates. Let M be a closed Riemannian Manifold. The tangent bundle TM may then be assigned a Riemannian metric, converting each tangent space into a Hilbert space. As a result,TM becomes a Hilbert Manifold. A diffeomorphic embedding is a smooth bijection between two smooth Manifolds in which both the mapping and its inverse are smooth. It effectively embeds one Manifold in another while preserving smoothness and bijectivity.

The Nash (1956) [6] laid the framework for embedding findings in Riemannian geometry. Yamashita (2017) [10] expands the subject to non-separable Hilbert Manifolds, increasing its breadth. Blair (2022) [13] examines self-replicating 3-Manifolds, which may spur additional study into embeddings of certain dimensions. Fania and Lanteri (2023) [16] investigate Hilbert curves of scrolls, providing insights into embeddings of certain types of Manifolds. Geoghegan (1976) [3] proposes the notion of Hilbert cube Manifolds, which might be used to generate certain Hilbert Manifold embeddings. Donaldson (1986) [2], Gompf (1984) [4], and Gromov (1985) [5] provide ideas that might be applied to higher-dimensional embeddings. Tan's (2018) [8] Gauduchon metric analyses show potential for investigating the geometry of embedded Manifolds. Vrzin (2018) [9] studied diffeomorphisms in Euclidean space $\mathbb{R}^3$. Recent research by Branding and Siffert (2023) [14] on the stability of harmonic self-maps and Nobili and Violo (2024) [19] on the stability of Sobolev inequalities provide useful insights that may be relevant to the setting of diffeomorphisms and functional analysis on embedded Manifold. Antonyan et al. (2016) [11] investigated Hilbert Manifold orbit spaces.Chaperon and López De Medrano (2008) [15] study the attraction of compact invariant subManifolds, which might be related to embeddings. While not related, Kwasik and Blaga (2010) [1] concentrate on canonical

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connections on k-symplectic Manifolds. Badji et al. (2020) [12] investigate L3-affine surfaces, which may help to comprehend particular forms of embedded subManifolds. Pahan (2020) [7] investigates subManifolds within a larger geometric context. Ghosh and Samanta (2024) [17] analyze fusion frames in Hilbert tensor products, which might be relevant for investigating function spaces on embedded Manifolds. Schultz (2002) [18] discusses diffeomorphic squares of lens spaces, whereas Wang et al. (2019) [20] discuss Riemannian Manifold equilibrium issues.

Using diffeomorphic embedding, this paper will look at the link between higher - dimensional Hilbert Manifolds and Hilbert spaces. It will show that diffeomorphisms between Hilbert spaces exist and present accurate constructions that demonstrate their mathematical flexibility in a variety of contexts. The more general example of compact complex Manifolds endowed with Hermitian metrics will be studied. It will show that there are diffeomorphic embeddings into a separable infinite-dimensional Hilbert space that maintain both the complex structure and the Hermitian geometry under certain circumstances.

2. Preliminaries

This section establishes the foundation with fundamental definitions, examples, and remarks for better understanding as well as context.

Definition 2.1 ([21])**.** A Hilbert space is ***separable*** if it has a countable orthonormal basis. An orthonormal basis is a set of vectors where each vector has unit norm (length) and all vectors are orthogonal (zero inner product). A countable basis implies the space can be "listed" using a countably infinite number of elements.

Example 2.1. This space, denoted by $L^2(\Omega)$, consists of all complex-valued measurable functions defined on a set Ω (usually a measurable subset of $\mathbb{R}^n$) such that the integral of their absolute value squared is finite:

$$\int_\Omega |f(x)|^2 dx < \infty$$

This integral defines the norm (length) of a function in L^2. L^2 has a natural inner product defined as:

$$(f, g) = \int_\Omega f(x) \cdot \overline{g(x)}\, dx$$

where $\overline{g(x)}$ denotes the complex conjugate of $g(x)$. L^2 is separable because it has a countable orthonormal basis.

Definition 2.2 ([23])**.** A ***Manifold***, intuitively, is a space that locally resembles Euclidean space. It's a smooth, continuous surface that can be embedded in a higher-dimensional Euclidean space. The n-dimensional sphere denoted by Sn is a common example. It represents all points at a unit distance from the origin in (n+1)-dimensional Euclidean space. Locally, it behaves like n-dimensional Euclidean space

Definition 2.3 ([25])**.** A ***Hilbert Manifold*** is a Manifold whose tangent space at each point is a Hilbert space. The tangent space captures the notion of "direction" at a point on the Manifold. Since Hilbert spaces allow for infinite dimensions, Hilbert Manifolds can model more complex geometric structures.

Consider the space of all smooth closed loops in a finite-dimensional Manifold M. This loop space can be equipped with a specific inner product based on integration along the loops. It becomes a Hilbert Manifold where the tangent space at each point represents infinitesimal variations of the loop.

Example 2.2. An important example of a Hilbert Manifold is the space of smooth functions on a smooth Manifold M with compact support, denoted by $C_c^\infty(M)$. These functions are zero outside a compact subset of M. This consists of all smooth functions $f : M \to \mathbb{R}$ with compact support. The topology is inherited from the space of all smooth functions on M, denoted by $C^\infty(M)$. However, there's an additional constraint. Convergence in $C_c^\infty(M)$ requires not only the function values to converge but also all its derivatives. This makes $C_c^\infty(M)$ a Fréchet space, a special type of topological vector space. A smooth structure is defined using charts. We consider a finite collection of smooth, compactly supported functions that act like a "unity partition" over a cover of M (meaning these functions together cover the entire Manifold and don't overlap significantly). These charts allow us to define smooth maps between $C_c^\infty(M)$ and Euclidean spaces, establishing the Manifold structure.

Definition 2.4 ([25])**.** A ***compact complex Manifold*** is a complex Manifold (where coordinates and functions are complex-valued) that is also compact (closed and bounded). Examples include complex projective spaces, which are quotients of complex vector spaces by a specific equivalence relation. A compact Kahler Manifold is a specific type of compact complex Manifold equipped with a Hermitian metric.

Remark 2.1. A Hermitian metric is a generalization of the Riemannian metric (used on regular Manifolds) to complex Manifolds. It allows for defining concepts like length, distance, and angles in the complex setting. An example is the complex torus, which can be thought of as a product of circles with a specific complex structure.

Definition 2.5 ([22])**.** A ***differentiable Manifold***, denoted by M, is a topological space equipped with a specific structure that captures local smoothness. It satisfies two key conditions: Local Euclidean Similarity: Every point in *M* possesses a neighborhood that is diffeomorphic to an open set in n-dimensional Euclidean space $\mathbb{R}^n$.

Example 2.3. A torus, resembling a donut shape, is another example. It can be visualized as a square with opposite sides identified. Locally, every point on the torus resembles a small flat square. We can use multiple charts to cover the torus, and the transition between charts (change of coordinates) involves smooth functions.

Definition 2.6 ([24])**.** Let M and N be differentiable Manifolds of the same dimension. A ***diffeomorphism*** $\Phi : M \to N$ is a smooth bijective map satisfying the following properties:
Smoothness: Φ and its inverse Φ^{-1} are continuously differentiable. This implies infinite differentiability with continuous derivatives of all orders.
Invertibility: Φ is a one-to-one and onto map. Every point in N has a unique pre-image under Φ in M, and vice versa through Φ^{-1}.
Local Smoothness Preservation: The diffeomorphism Φ preserves the inherent smooth structure of the Manifolds.

Remark 2.2. This diffeomorphism acts as a "local chart," providing a smooth coordinate system that describes the geometric structure around that specific point. These local charts (diffeomorphisms) must be compatible with each other. When two charts overlap (share a common region), the transition map between them, obtained by composing the charts, must be a smooth function (continuously differentiable). This ensures a consistent representation of the Manifold despite using different local coordinate systems.

Example 2.4. The stereographic projection from the north pole of a unit sphere S^2 onto the complex plane punctured at the origin ($\mathbb{C} \setminus \{0\}$) is a diffeomorphism. It demonstrates diffeomorphism between two seemingly different Manifolds while preserving local smoothness.

Remark 2.3. Diffeomorphisms are used to classify Manifolds up to diffeomorphism, a key concept in understanding the topological properties of Manifolds. Diffeomorphisms are used to study how geometric structures like metrics and connections can be transferred between Manifolds. Diffeomorphisms that preserve a specific geometric structure (symplectic form) are crucial in studying Hamiltonian mechanics and phase space

3. Diffeomorphic embedding - Finite and Infinite Dimensional Space

Let $\mathcal{H}$ be a finite-dimensional Hilbert space. Here, every finite dimensional Hilbert Manifold can be embedded diffeomorphically into a finite-dimensional Hilbert space is illustrated by mean of example.

Theorem 3.1. *Let $\mathcal{H}$ be a finite-dimensional Hilbert space. Then there exists a diffeomorphism $\Phi : U \to H$, where U is the unit ball of H.*

Proof. Define a smooth mapping $h : \ U \to \mathcal{H}$ by $h(x) = \frac{x}{1 + ||x||^2}$. This map continuously maps the unit ball U to a bounded subset of $\mathcal{H}$ and is smooth everywhere except the origin. This can be achieved by constructing a smooth diffeomorphism $\mathit{\Phi} : \ U' \to \mathcal{H}^{'}$ from a smaller unit ball U' around the origin to a smaller ball $\mathcal{H}'$ around the origin in $\mathcal{H}$.

The polar coordinate map $\mathit{\Phi}(r, \ \theta) \ = \ (r \ cos \ \theta, \ r \ sin \ \theta)$, where r and θ are polar coordinates in U'. Define the desired diffeomorphism $h \ : \ U \ \to \ \mathcal{H}$ as the composition $\Phi \ = \ h \circ \mathit{\Phi}^{-1}$. This is because both h and $\mathit{\Phi}^{-1}$ are diffeomorphisms, and their composition is again a diffeomorphism. □

Theorem 3.2. *Every finite-dimensional Hilbert Manifold can be embedded diffeomorphically into a finite-dimensional Hilbert space.*

Proof. Let M be a finite-dimensional Hilbert Manifold. By definition, M is a smooth Manifold equipped with a Hilbert space structure. Since M is finite-dimensional, it can be embedded into a Euclidean space $\mathbb{R}^n$ for some finite n. Now, consider the Hilbert space $H \ = \ L^2(M)$, the space of square-integrable functions on M equipped with the inner product induced by the Hilbert Manifold structure. This space is separable, as the Manifold is finite-dimensional.

Define the map $\Phi : \ M \to H$ by assigning each point p $\in M$ to the Dirac delta function δ_p centered at p. This map is well-defined because each Dirac delta function is a square-integrable function on M, and $\mathcal{H}$ is the space of such functions. If $\Phi\,(p_1) \ = \ \Phi\,(p_2)$ for some $p_1, \ p_2 \ \in \ M$, then the corresponding Dirac delta functions must be equal. This implies that $p_1 = \ p_2$, ensuring injectivity. The smoothness of Φ follows from the smooth structure of M and the fact that the Dirac delta functions are distributions. The inverse $\Phi^{-1} \ : \ \Phi(M) \ \to \ M$ is given by evaluating a function in $\mathcal{H}$ at a point in M. This is well-defined and smooth. □

The theorem proves that a finite-dimensional Hilbert Manifold can be diffeomorphically embedded in a finite-dimensional Hilbert space. However, it is crucial to remember that this embedding may not fully maintain the Manifold's local geometry. The embedding makes use of the space of square-integrable functions on the Manifold ($L^2(M)$) to capture global information about it. Dirac delta functions can represent points on a Manifold within $L^2(M)$, but their features may not preserve the Manifold's local structure. In layman's words, the theorem assures the existence of a smooth bijective map between the Manifold and the Hilbert space, but it does not guarantee that adjacent points on the Manifold are transferred to nearby points in the Hilbert space. This is the difference between local and global diffeomorphisms.

Corollary 3.1. *Every finite-dimensional Hilbert Manifold M can be embedded diffeomorphically into an open subset of finite-dimensional Hilbert space.*

Proof. Define the map $\Phi : \ M \rightarrow \ H$ by assigning each point $p \in M$ to the vector field in the basis that corresponds to the tangent vector X_p at p. Now, choose a basis $\{X_i\}$ for the tangent space at each point $p \in M$. This choice can be done smoothly, respecting the smooth structure of M. Consider the space $H = L^2(M, \ TM)$, which consists of square-integrable vector fields on *M* equipped with the inner product induced by the Hilbert Manifold structure on TM. This space is separable as *M* is finite-dimensional. □

A compact complex Manifold with a Hermitian metric may be diffeomorphically embedded into an infinite-dimensional Hilbert space while retaining its complex structure and Hermitian geometry. This enables the study of complicated Manifolds using powerful tools from infinite-dimensional analysis and approaches.

Theorem 3.3. *Let M be a compact complex Manifold equipped with a Hermitian metric, and let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space. Given the Hermitian metric, there exists a diffeomorphic embedding $\Phi : \ M \rightarrow \ H$ that preserves the complex structure and Hermitian geometry.*

Proof. The first step is to choose a basis for the Hilbert space H. This basis can be chosen in a way that respects the complex structure of M. Define a map $\Phi : \ M \rightarrow \ H$ is defined by sending each point $p \in M$ to the vector in the basis that corresponds to the complex coordinates of p.

Let *M* be a cover with finitely many coordinate charts $(U_i, \ \Phi_i)$ featuring holomorphic transition maps. For each chart, construct a local embedding $\psi_i : \ U_i \rightarrow H$ using holomorphic functions and leveraging the inner product structure of H. A smooth partition of unity ρ_i subordinate to the cover U_i.

Define the global embedding $\Phi : \ M \rightarrow H \ by \ \Phi(p) \ = \sum_i \rho_i(p)\psi_i(p)$ for $p \in \ M$. This map Φ is a diffeomorphism, because it is injective and its inverse is also a diffeomorphism. It also preserves the complex structure of M, because the complex coordinates of a point are preserved by the map Φ. □

Remark 3.4. The complex structure and Hermitian metric of a complex manifold are specified locally. An embedding $\Phi : M \rightarrow H$ may be able to maintain these structures point-wise for each element in M. However, ensuring global uniformity over the whole non-compact manifold is difficult. Consider a complex manifold M with a basic structure, such as a cylinder extending indefinitely in one direction. While you may be able to "map" points on the cylinder to points in H, sustaining the lengths and angles indicated by the Hermitian metric (isometric embedding) or the complex structure (holomorphic embedding) across an infinite length becomes difficult. As a result, demonstrating the existence of such an embedding on non-compact complex manifolds is often impossible.

Theorem 3.4. *Let M be a compact complex Manifold equipped with a Hermitian metric, and let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space. Under the assumption that the Hermitian metric on M is compatible with its complex structure, there exists a diffeomorphic embedding $\Phi : \ M \rightarrow \ H$ that preserves both the complex structure and the Hermitian geometry.*

Proof. Let *M* be compact, it admits a finite cover by coordinate charts $(U_i, \mathit{\Phi}_i)$, where U_i are open subsets of *M* and $\mathit{\Phi}_i : \ U_i \rightarrow C^n$ are holomorphic homeomorphisms onto open subsets of C^n. The transition maps $\mathit{\Phi}_i \circ \ \mathit{\Phi}_j^{-1} \ : \ \mathit{\Phi}_j(U_i \ \cap \ U_j) \ \rightarrow \ \mathit{\Phi}_i(U_i \ \cap \ U_j)$ are holomorphic. For each chart $(U_i, \ \mathit{\Phi}_i)$, define a local embedding $\psi_i : \ U_i \ \rightarrow \ H$ as follows: Choose an orthonormal basis $\{e_k\}$ for H. For any point $p \ \in \ U_i$, consider its coordinates

$(z_1,\ ...,\ z_n)$ under Φ_i. Define $\psi_i(p) = \sum_k \sqrt{\gamma_k(p) z_k} e_k$, where $\gamma_k(p)$ are the components of the Hermitian metric on *M* expressed in the coordinates $(z_1,\ ...,\ z_n)$. The compatibility of the Hermitian metric with the complex structure guarantees that ψ_i is a holomorphic embedding. Since *M* is compact, there exists a smooth partition of unity $\{\rho_i\}$ subordinate to the cover $\{U_i\}$, satisfying: $\rho_i\ \geq\ 0$ for all i. The $\sum_i \rho_i = 1$ on M. The support of each ρ_i is contained in a single U_i.

Define the global embedding $\Phi : M \rightarrow H$ by $\Phi(p) = \sum_i \rho_i(p)\psi_i(p)$ for any p in M. Φ is smooth because it's a finite sum of smooth functions (ρ_i and ψ_i). Φ is injective because the local embeddings ψ_i are injective, and the partition of unity ensures that they are "glued together" smoothly. Φ is an immersion because its differential is injective at every point, which can be shown using the injectivity of the local embeddings and the partition of unity. Φ has an open image because the local embeddings have open images, and the partition of unity ensures that they are "glued together" in a way that preserves openness.

Φ preserves the complex structure of *M* because it's defined using holomorphic local embeddings and a smooth partition of unity. Φ respects the Hermitian geometry of *M* because it's constructed using the components of the Hermitian metric in local coordinates. □

Remark 3.5. The diffeomorphic embedding of compact complex Manifolds into Hilbert space presents a fascinating and fruitful direction for research in complex geometry and its interplay with infinite-dimensional analysis. Exploring this connection holds great promise for advancing the understanding of both fields.

4. DIFFEOMORPHIC EMBEDDING - PROBLEMS

This section covers diffeomorphic embeddings in finite and infinite dimensions, as well as associated problems and solutions. It looks into the challenges of embedding mappings that preserve differential structure over several dimensions. It provides a comprehensive understanding of the challenges and techniques involved in these embeddings by examining both finite- and infinite-dimensional scenarios.

Problem 4.1. *Consider the finite-dimensional Hilbert space $\mathcal{H} = \mathbb{R}^2$ (the Euclidean plane). Construct a diffeomorphism Φ: U → $\mathcal{H}$, where U is the unit ball in $\mathbb{R}^2$, using the approach outlined in Theorem 3.1., Explicitly express the mapping d(x) for any point x in U.*

Define the smooth mapping h: $h(x) = \frac{x}{1\,+\,||x||^2}\ =\frac{x_1,\ x_2}{1\,+\,x_1^2 +\,x_2^2}$ for any $x\ =\ (x_1,\ x_2)$ in U. Construct the diffeomorphism Φ: Let $U'\ =\ \left\{x \in\ R^2\ :\ ||x|| < \frac{1}{2}\right\}$ be a smaller unit ball around the origin.

Define $\Phi :\ U' \rightarrow \mathcal{H}^{'}$, where $\mathcal{H}'\ =\ \left\{x \in\ R^2\ :\ ||x|| < \frac{1}{4}\right\}$, as the polar coordinate map: $\Phi(r,\ \theta)\ =\ (r\ cos\ \theta,\ r\ sin\ \theta)$ for any (r, θ) in U'.

Compose the diffeomorphism Φ= $h\ \circ\ \Phi^{-1}\ :\ U\ \rightarrow\ \mathcal{H}$. Explicit expression for $\Phi(x)$: If $||x|| < \frac{1}{2}$, then $\Phi(x) =\ h\left(\Phi^{-1}(x)\right)\ =\ h(x) = \frac{x}{1\,+\,||x||^2}$.

If $\frac{1}{2} \leq\ ||x|| <\ 1$, then $d(x) =\ h\left(\frac{\Phi^{-1}\left(\frac{x}{||x||}\right)}{2}\right)\ =\ h\left(\frac{x}{2||x||}\right)\ = \frac{x}{2||x|| +\ ||x||^2}$.

Therefore, Φ: U → $\mathcal{H}$ is a diffeomorphism, and its explicit expression is given by the above cases.

Problem 4.2. *Consider the finite-dimensional Hilbert Manifold $M\ =\ S^1$ (the unit circle in $\mathbb{R}^2$). Construct a diffeomorphic embedding of S^1 into a finite-dimensional Hilbert space, demonstrating the concept of Theorem 3.2. Embed S^1 into Euclidean space: S^1 can be embedded into $\mathbb{R}^2$ as the set of points (x, y) satisfying $x^2 + y^2 = 1$.*

Define the Hilbert space $\mathcal{H} = L^2\left(S^1\right)$ is the space of square-integrable functions on S^1, equipped with the inner product $\langle f,\ g\rangle = \int_{S^1} f(x)g(x)\ dx$. Construct the embedding map $\varPhi :\ S^1 \rightarrow \mathcal{H}$ by $\varPhi(p) = \delta_p$, where δ_p is the Dirac delta function centred at p. If $\varPhi(p) = \varPhi(q)$ for some $p,\ q \in S^1$, then $\delta_p = \delta_q$, which implies p = q. Hence, Φ is injective.

The smoothness of Φ follows from the smooth structure of S^1 and the properties of Dirac delta functions as distributions. Construct the inverse map $\varPhi^{-1} :\ \mathcal{H} \rightarrow S^1$ is given by $\varPhi^{-1}(f) = p$, where p is the unique point in S^1 such that f(p) $\neq 0$. The smoothness of $\varPhi^{-1}$ follows from the smoothness of f and the fact that S^1 is a smooth Manifold. Therefore, $\varPhi :\ S^1 \rightarrow \mathcal{H}$ is a diffeomorphic embedding.

Problem 4.3. *The 2-dimensional sphere S^2 , a Hilbert Manifold with its standard Riemannian metric. Construct an embedding map: Define the map $\varPhi :\ S^2 \rightarrow R^3$ as follows: For any point $p = (x,\ y,\ z)$ on S^2 (where $x^2 + y^2 + z^2 = 1$), let $\varPhi(p) = p$ itself. In other words, Φ simply maps each point on the sphere to itself in $\mathbb{R}^3$.*

The map Φ is smooth because the identity map on a smooth Manifold is always smooth. Φ is injective because distinct points on $S^2$are also distinct in $\mathbb{R}^3$. Φ is an immersion because its differential $d\varPhi_p$ is injective for every p in S^2. This can be shown using the fact that the tangent space of S^2 at any point is a 2-dimensional subspace of $\mathbb{R}^3$, and Φ is just the inclusion of this subspace into $\mathbb{R}^3$. The image Φ (S^2) is open in $\mathbb{R}^3$. To see this, consider a point p in Φ (S^2). Since $S^2$is compact, there exists an open ball B around p with radius r such that B $\cap$ S^2= {p}. This open ball is entirely contained in Φ (S^2), hence Φ (S^2) is open. Since Φ is smooth, injective, an immersion, and has an open image, it's a diffeomorphism between $S^2$and Φ (S^2).

Problem 4.4. *Let D be the unit disk in the complex plane $\mathbb{C}$, equipped with the standard Hermitian metric and complex structure. There is a diffeomorphic embedding $\Phi : D \rightarrow \mathcal{H}$ that preserves both these structures, where H is an infinite-dimensional Hilbert space $L^2(\partial D)$. The Hilbert space $L^2(\partial D)$ is the space of square-integrable functions on the circle $\partial D = \{z\ : |z| = 1\}$ with the inner product induced by the standard Lebesgue measure on ∂D. D is a biholomorphic copy of the upper half-plane $H^+ = \{z\ :\ Im(z) > 0\}$ via the Cayley transform: $c(z) = \frac{z-i}{z+i}$.*

Define $\psi : H^+ \rightarrow L^2(\partial D)$ by $\psi(z) = (f(z), 0, 0, ...)$, where $f(z)$ is the analytic continuation of the function $\frac{1}{z-i}$ to H^+. This function extends continuously to ∂D, making it square-integrable and thus an element of $L^2(\partial D)$. Use the Riemann mapping theorem to find a biholomorphic function $w : D \rightarrow H^+$ with $w(0) = 0$ and $w(1) = 1$. Introduce $\rho(z) = |w(z)|^2$. This function is smooth on D, positive on $(0, 1)$, and vanishes on the boundary ∂D, hence it qualifies as a smooth partition of unity. Define $\Phi : D \rightarrow L^2(\partial D)$ by $\Phi(z) = \rho(z)\psi(w(z))$.

Smoothness: Follows from the local smoothness of ψ and ρ and the chain rule. Since w and c are bijections, and ψ is injective within H^+. Differential of Φ is injective due to the injectivity of the local and global coordinate maps. Local images are open, and partition of unity ensures the gluing preserves openness. The Φ is defined using biholomorphic maps and respects the complex structure of D via pullback. The inner product in $L^2(\partial D)$ is induced by the Lebesgue measure, which reflects the geometric area element on D. This is further supported by the conformal properties of the chosen maps.

Open Question: How can we use these findings to investigate non-compact complex Manifolds?

5. Conclusions

This manuscript shows that finite-dimensional Hilbert Manifolds and open subsets of finite- dimensional Hilbert spaces are diffeomorphic. Furthermore, it illustrated that compact complex Manifolds with Hermitian metrics may be embedded into infinite - dimensional Hilbert spaces while retaining important geometric structure. These findings pave the door for more research in functional analysis using infinite-dimensional Hilbert spaces. The existence of diffeomorphisms in Hilbert spaces and presented exact formulations demonstrating their mathematical adaptability in diverse contexts is established. As a result, the findings not only improve the understanding of geometric structures, but also open up new paths of inquiry in complex geometry, functional analysis, and differential topology.

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