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Some fixed point results for extended P_{p_Θ}-contractions via extended wt-distance and its applications

IREM EROĞLU

ABSTRACT. In this paper, we introduce the notion of extended wt-distance which is a generalization of wt-distance and we obtain some fixed point theorems for extended P_{p_Θ}-contractions via extended wt-distance over partially ordered complete extended b-metric spaces. Moreover, we prove some fixed point results for the cyclic case. Finally, we give an application of our main result.

1. INTRODUCTION

Fixed point theory gained recognition with the "Banach Contraction Principle" proved by Banach in 1922 and has been studied by many researchers. The notion of b-metric was defined by Bakhtin [3]. Bakhtin [3] extended Banach Fixed Point Theorem to the concept of complete b-metric spaces. In [12], Karapınar et al. proved a Caristi type theorems in b-metric spaces. Then, Romaguera [14] answered the question which is asked by Karapınar et al [12] in the negative direction. Moreover, Romaguera [14] introduced the $\preceq$-correlation notion. Kamran et al. [11] introduced the definition of extended b-metric space and they proved the Banach Contraction Theorem for extended b-metric spaces. In [7], Guran et al. obtained a generalization of the Banach Contraction principle by using the notion of cyclic contraction. The mentioned result is as follows:

Theorem 1.1. [7] *Let (X, d_Θ) be a complete extended b-metric space with d_Θ a continuous functional. Let $\{A_i\}_{i=1}^t$, where t is a positive integer, be nonempty closed subsets of X and suppose $T : \cup_{i=1}^t A_i \to \cup_{i=1}^t A_i$ is a cyclic operator that satisfies the following conditions:*
(i) $T(A_i) \subseteq A_{i+1}$, for all $i \in \{1, 2, .., t\}$
(ii) $d_\Theta(Tx, Ty) \leq \lambda d_\Theta(x, y)$ for all $x \in A_i, y \in A_{i+1}$ where $\lambda \in [0, 1)$ be such that for each $x \in X$, $\lim_{m,n\to\infty} \Theta(x_m, x_n) < \frac{1}{\lambda}$ where $x_n = T^n(x_0)$, $n = 1, 2, ...$
Then, T has a unique fixed point z. Moreover, for each $y \in X, T^n y \to z$.

In [8], the existence of fixed points of rational type contractions in the setting of extended b-metric spaces has been proved. Samreen et al. [15] introduced a new class of comparison functions and proved some fixed point theorems. Moreover, they applied the results to integral equations. The reader can see more work about the extended b-metric spaces in [5, 1].

The concept of w-distance was introduced by Kada et al. [10] and they gave a fixed point theorem in a complete metric space which is a general form of Ćirić' s fixed point theorem.

In [9], the notion of wt-distance has been introduced as follows:

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Definition 1.1. [9] *Let (X, d, s) be a b-metric space with $s \geq 1$. Then a function $p : X \times X \rightarrow [0, \infty)$ is called a wt-distance on X if the following are satisfied:*

(i) $p(x, z) \leq s[p(x, y) + p(y, z)]$
(ii) for any $x \in X$, $p(x, .) : X \rightarrow [0, \infty)$ is s-lower semicontinuous
(iii) for any $\epsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \epsilon$.

The s-lower semi-continuity concept which is given in the definition means that, if either $\lim_{x_n \to x_0} f(x_n) = \infty$ or $f(x_0) \leq \lim_{x_n \to x_0} sf(x_n)$, whenever $x_n \in X$ for each $n \in \mathbb{N}$ and $x_n \to x_0$. In [6], they proved some fixed point theorems for contractive type multivalued operators by using wt-distance. Very recently, Romaguera [13] obtained a generalization of b-metric version of Matkowski's theorem by using wt-distance.

Altun et al. [2] defined the P_w-contraction mapping by using the w-distance and they proved the following fixed point theorem:

Theorem 1.2. [2] *Let (X, d) be a complete metric space, w be a w-distance on X and $T : X \rightarrow X$ be a P_w-contraction. Assume that one following hold:*

(i) T is continuous
(ii) w is continuous
(iii) for every $y \in X$ with $y \neq Ty$

$$inf\{w(x, y) + w(x, Tx) : x \in X\}$$

Then, T has a unique fixed point $z \in X$ such that $w(z, z) = 0$.

In this paper, we introduce the notion of extended wt-distance which is a generalization of wt-distance. Then, we will define P_{p_θ}-contraction mappings defined on partially ordered extended b-metric spaces. Moreover, we extend Theorem 1.2 to the concept of partially ordered extended b-metric spaces by using extended wt-distance. Furthermore, we will give fixed point theorems for cyclic-type operators and we extend Theorem 1.1 to partially ordered extended b-metric spaces through the use of extended wt-distance. Finally, we give an application of our main result to Volterra-Fredholm type integral equation.

2. Extended wt-distance

This section begins by defining the concept of extended wt-distance, which is a generalization of wt-distance.

Definition 2.2. *Let (X, d_Θ) be an extended b-metric space, where Θ is a function such that $\Theta : X \times X \rightarrow [1, \infty)$. The extended wt-distance can be described as a function $p_\Theta : X \times X \rightarrow [0, \infty)$, if the following conditions are satisfied:*

(i) $p_\Theta(x, z) \leq \Theta(x, z)[p_\Theta(x, y) + p_\Theta(y, z)]$
(ii) for any $x \in X$, $p_\Theta(x, .) : X \rightarrow [0, \infty)$ is Θ-lower semi-continuous
(iii) for any $\epsilon > 0$, there exists $\delta > 0$ such that $p_\Theta(z, x) \leq \delta$ and $p_\Theta(z, y) \leq \delta$ imply $d_\Theta(x, y) \leq \epsilon$

We mean by the notion of Θ-lower semi-continuity given in the assumption (ii): if either $\lim_{x_n \to x_0} p_\Theta(x, x_n) = \infty$ or $p_\Theta(x, x_0) \leq \liminf_{x_n \to x_0} \Theta(x, x_0) p_\Theta(x, x_n)$, whenever $x_n \in X$ for each $n \in \mathbb{N}$ and $x_n \to x_0$ according to τ_{d_Θ} .

Remark 2.1. *If we take $\Theta(x, y) = s \geq 1$ in the definition of extended wt-distance we obtain the definition of wt-distance, but if d_Θ is an extended b-metric on X, then d_Θ is not necessarily an extended wt-distance on the extended b-metric (X, d_Θ).*

Remark 2.2. *It is obvious that if the function* Θ *satisfies* $sup_{x,y\in X}\Theta(x,y) < \infty$, *then every extended b-metric function* d_Θ *on* X *is an extended wt-distance on* X.

Now we give some extended wt-distance examples.

Example 2.1. *Let* $X = [0,\infty)$. *Define the mappings* $\Theta : X \times X \to [1,\infty)$ *and* $d_\Theta : X \times X \to [0,\infty)$ *as follows:* $\Theta(x,y) = 1 + x + y$ *and*

$$d_\Theta(x,y) = \begin{cases} x^2 + y^2, & x, y \in X, x \neq y \\ 0, & x = y \end{cases}$$

Then (X, d_Θ) *is an extended b-metric space (see* [8] *). Let consider the mapping* $p_\Theta : X \times X \to [0,\infty)$ *defined by* $p_\Theta(x,y) = y^2$. *Then,* p_Θ *is an extended wt-distance on* (X, d_Θ).

Example 2.2. *Let* $X = [0,\infty)$. *Define the mappings* $\Theta : X \times X \to [1,\infty)$ *and* $d_\Theta : X \times X \to [0,\infty)$ *as follows:* $\Theta(x,y) = 1 + x + y$ *and*

$$d_\Theta(x,y) = \begin{cases} x + y, & x, y \in X, x \neq y \\ 0, & x = y \end{cases}$$

Then (X, d_Θ) *is an extended b-metric space (see* [8] *). Let consider the mapping* $p_\Theta : X \times X \to [0,\infty)$ *defined by* $p_\Theta(x,y) = y$. *It is clear that* p_Θ *is an extended wt-distance on* (X, d_Θ).

We now give the following lemma, which is primary for our results. The proof is quite similar to [10] and we omit it.

Lemma 2.1. *Let* (X, d_Θ) *be an extended b-metric space and* p_Θ *be an extended wt-distance on* X. *Let* (x_n) *and* (y_n) *be sequences in* X, *let* (α_n) *and* (β_n) *be sequences in* $[0,\infty)$ *converging to zero and let* $x, y, z \in X$. *Then, the following hold:*

(1) If $p_\Theta(x_n, y) \leq \alpha_n$ *and* $p_\Theta(x_n, z) \leq \beta_n$ *for any* $n \in \mathbb{N}$, *then* $y = z$. *In particular, if* $p_\Theta(x,y) = 0$ *and* $p_\Theta(x,z) = 0$, *then* $y = z$.
(2) If $p_\Theta(x_n, y_n) \leq \alpha_n$ *and* $p_\Theta(x_n, z) \leq \beta_n$ *for any* $n \in \mathbb{N}$, *then* $d_\Theta(y_n, z) \to 0$
(3) If $p_\Theta(x_n, x_m) \leq \alpha_n$ *for any* $n, m \in \mathbb{N}$ *with* $m > n$, *then* (x_n) *is a Cauchy sequence.*
(4) If $p_\Theta(y, x_n) \leq \alpha_n$ *for any* $n \in \mathbb{N}$, *then* (x_n) *is a Cauchy sequence.*

3. Main result

In this section, we present the term of extended P_{p_Θ}-contraction which is a generalization of P_w-contraction mapping given in [2]. Throughout of this paper, τ_{d_Θ} is the topology which is generated by the extended b-metric d_Θ and we will denote by $\Gamma((X, \tau_{d_\Theta}))$ the set of all accumulation points. Here, accumulation point means that all points $x \in X$ such that there is a sequence of X that τ_{d_Θ}-converges to x.

Definition 3.3. *Let* $(X, d_\Theta, \preceq)$ *be a partially ordered extended b-metric space, where* d_Θ *is an extended b-metric and* $\preceq$ *is a partial order such that reflexive, transitive and antisymmetric relation on* X. *Let* p_Θ *be an extended wt-distance on* X *and* $T : X \to X$ *be a mapping. If there exists a nonnegative real number* $\kappa < 1$ *satisfying*

$$p_\Theta(Tx, Ty) \leq \kappa[p_\Theta(x,y) + \mid p_\Theta(x, Tx) - p_\Theta(y, Ty) \mid] \tag{3.1}$$

for all $x \preceq y$ *or* $y \in \Gamma((X, \tau_{d_\Theta}))$. *Then* T *is said to be extended* P_{p_Θ}*-contraction mapping.*

Now we give our main result of this section:

Theorem 3.3. *Let* $(X, d_\Theta, \preceq)$ *be a partially ordered complete extended b-metric space and* p_Θ *be an extended wt-distance on* X *and* d_Θ *continuous functional. Let* $T : X \to X$ *be an increasing* P_{p_Θ}*-contraction mapping with* $\lim\limits_{m,n\to\infty} \Theta(x_n, y_m) < \dfrac{1}{\kappa}$, *where* $x_n = T^n(x_0)$, *for any* $x_0 \in$

X and $\kappa \in [0,1)$ and (y_m) be any sequence in X. If there exists a point $x_0 \in X$ with $x_0 \preceq Tx_0$ and one of the conditions listed below is true:

(i) T is continuous
(ii) p_Θ is continuous
(iii) for every $y \in X$ with $y \neq Ty$ and $x \preceq Tx$

$$inf\{p_\Theta(x,y) + p_\Theta(x,Tx) : x \in X\} > 0 \tag{3.2}$$

Then, T has a fixed point.

Proof. Let $x_0 \in X$ with $x_0 \preceq Tx_0$. Take a look at the Picard sequence (x_n) that is associated with $x_{n+1} = Tx_n$ for $n \geq 0$, where T is increasing and by induction we have $x_0 \preceq Tx_0 \preceq T^2x_0 \preceq ... \preceq T^n x_0 \preceq T^{n+1} \preceq ...$
Now we consider the two cases given below:

Case I: Suppose that there exists $n_0 \in \mathbb{N}$ such that $p_\Theta(x_{n_0}, x_{n_0+1}) = 0$. Then, we have that $p_\Theta(x_{n_0+1}, x_{n_0+2}) = 0$. Indeed,

$$\begin{aligned} p_\Theta(x_{n_0+1}, x_{n_0+2}) &\leq \kappa[p_\Theta(x_{n_0}, x_{n_0+1}) + \mid p_\Theta(x_{n_0}, Tx_{n_0}) - p_\Theta(x_{n_0+1}, Tx_{n_0+1}) \mid] \\ &= \kappa[p_\Theta(x_{n_0}, x_{n_0+1}) + \mid p_\Theta(x_{n_0}, x_{n_0+1}) - p_\Theta(x_{n_0+1}, x_{n_0+2}) \mid] \\ &= \kappa p_\Theta(x_{n_0+1}, x_{n_0+2}) \end{aligned} \tag{3.3}$$

Since $\kappa < 1$, we have that $p_\Theta(x_{n_0+1}, x_{n_0+2}) = 0$ and by the condition (i) of the extended wt-distance we have

$$p_\Theta(x_{n_0}, x_{n_0+2}) \leq \Theta(x_{n_0}, x_{n_0+2})[p_\Theta(x_{n_0}, x_{n_0+1}) + p_\Theta(x_{n_0+1}, x_{n_0+2})] = 0 \tag{3.4}$$

Now we have that $p_\Theta(x_{n_0}, x_{n_0+1}) = 0$ and $p_\Theta(x_{n_0}, x_{n_0+2}) = 0$. By Lemma 2.1 we get that $x_{n_0+1} = x_{n_0+2} = Tx_{n_0+1}$. Then, x_{n_0+1} is a fixed point.

Case II: Now suppose that $p_\Theta(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$. Then we have

$$\begin{aligned} p_\Theta(x_{n+1}, x_{n+2}) &= p_\Theta(Tx_n, Tx_{n+1}) \\ &\leq \kappa[p_\Theta(x_n, x_{n+1}) + \mid p_\Theta(x_n, Tx_n) - p_\Theta(x_{n+1}, Tx_{n+1}) \mid] \\ &= \kappa[p_\Theta(x_n, x_{n+1}) + \mid p_\Theta(x_n, x_{n+1}) - p_\Theta(x_{n+1}, x_{n+2}) \mid] \end{aligned} \tag{3.5}$$

This case necessitates that $p_\Theta(x_{n+1}, x_{n+2}) < p_\Theta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Hence from (3.5), we have that

$$p_\Theta(x_{n+1}, x_{n+2}) \leq \frac{2\kappa}{1+\kappa} p_\Theta(x_n, x_{n+1}) \tag{3.6}$$

for all $n \in \mathbb{N}$. Therefore we get $p_\Theta(x_{n+1}, x_{n+2}) \leq \lambda^{n+1} p_\Theta(x_0, x_1)$ for all $n \in \mathbb{N}$, where $\lambda = \dfrac{2\kappa}{\kappa+1} < 1$.

It is now possible to demonstrate that (x_n) is a Cauchy sequence. For any $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} p_\Theta(x_n, x_m) &\leq \Theta(x_n, x_m)\lambda^n p_\Theta(x_0, x_1) + \Theta(x_n, x_m)\lambda^{n+1} p_\Theta(x_0, x_1) \\ &+ \Theta(x_n, x_m)\Theta(x_{n+1}, x_m)...\Theta(x_{m-1}, x_m)\lambda^{m-1} p_\Theta(x_0, x_1) \\ &\leq p_\Theta(x_0, x_1)[\Theta(x_1, x_m)\Theta(x_2, x_m)...\Theta(x_{n-1}, x_m)\lambda^n + \\ &+ \Theta(x_1, x_m)\Theta(x_2, x_m)...\Theta(x_n, x_m)\Theta(x_{n+1}, x_m)\lambda^{n+1} + ... \\ &+ \Theta(x_1, x_m)\Theta(x_2, x_m)...\Theta(x_n, x_m)\Theta(x_{n+1}, x_m)\Theta(x_{m-1}, x_m)\lambda^{m-1} \end{aligned}$$

Since $\lim\limits_{m,n\to\infty} \Theta(x_{n+1}, x_m) < \dfrac{1}{\kappa}$, the series $\sum_{n=1}^{\infty} \lambda^n \prod_{r=1}^{n} \Theta(x_r, x_m)$ converges by ratio test for each $m \in \mathbb{N}$.

Let $\Lambda = \sum_{n=1}^{\infty} \lambda^n \prod_{r=1}^{n} \Theta(x_r, x_m)$ and $\Lambda_n = \sum_{n=1}^{n} \lambda^n \prod_{r=1}^{n} \Theta(x_r, x_m)$. Thus we have for $n < m$, $p_\Theta(x_n, x_m) \leq p_\Theta(x_0, x_1)[\Lambda_{m-1} - \Lambda_n]$, so from Lemma 2.1, we get that (x_n) is a Cauchy sequence.

Since X is complete, there exists $z \in X$ such that (x_n) is converging according to τ_{d_Θ}. From the Θ-lower semicontinuity of p_Θ, we have the following:

$$\begin{aligned} p_\Theta(x_n, z) &\leq \liminf_{m\to\infty} \Theta(x_n, z) p_\Theta(x_n, x_m) \\ &\leq \Theta(x_n, z) p_\Theta(x_0, x_1)[\Lambda_{m-1} - \Lambda_n] \end{aligned} \tag{3.7}$$

Now if T is continuous, then $x_{n+1} = Tx_n \to Tz$ and by the uniqueness of the limit we get that $z = Tz$. Moreover, $p_\Theta(z, z) = 0$. Indeed, since $z \preceq z$ by the contraction condition we have

$$p_\Theta(Tz, Tz) \leq \kappa[p_\Theta(z, z)+ \mid p_\Theta(z, Tz) - p_\Theta(z, z) \mid]$$

The fact that $p_\Theta(z, z) \leq \kappa p_\Theta(z, z)$ is a contradiction, except for when $p_\Theta(z, z) = 0$. Now, let us assume p_Θ is continuous, then we obtain

$$\lim_{n\to\infty} p_\Theta(x_n, z) \leq \lim_{n\to\infty} \Theta(x_n, z) p_\Theta(x_0, x_1)[\Lambda_{m-1} - \Lambda_n] \tag{3.8}$$

and $\lim_{n\to\infty} p_\Theta(x_n, z) = 0$. Moreover, $\lim_{m,n\to\infty} p_\Theta(x_n, x_m) = p_\Theta(z, z) = 0$.

Now since z is an accumulation point of X, putting $x = x_n$ and $y = z$ in the contraction condition it is possible for us to write the following equation:

$$p_\Theta(x_{n+1}, Tz) \leq \kappa[p_\Theta(x_n, z)+ \mid p_\Theta(x_n, x_{n+1}) - p_\Theta(z, Tz)] \tag{3.9}$$

for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ and using the continuity of p_Θ, we get that

$$p_\Theta(z, Tz) \leq \kappa p_\Theta(z, Tz)$$

which is a contradiction. Thus, $p_\Theta(z, Tz) = 0$. Since we have $p_\Theta(z, z) = p_\Theta(z, Tz) = 0$ and from Lemma 2.1 we have $z = Tz$.

Finally, assume that condition (iii) holds and $z \neq Tz$. Then, for all $x \preceq Tx$ we have

$$0 < inf\{p_\Theta(x, z) + p_\Theta(x, Tx) : x \in X\} \tag{3.10}$$

Since $x_n \preceq x_{n+1} = Tx_n$, we get

$$\begin{aligned} 0 &< inf\{p_\Theta(x_n, z) + p_\Theta(x_n, Tx_n) : x \in X\} \\ &= inf\{p_\Theta(x_n, z) + p_\Theta(x_n, x_{n+1}) : n \in \mathbb{N}\} \to 0, \text{ as } n \to \infty. \end{aligned}$$

which is a contradiction, thus $z = Tz$. Moreover since $z \preceq z$ by the contraction condition we have

$$p_\Theta(Tz, Tz) = p_\Theta(z, z) \leq \kappa[p_\Theta(z, z)+ \mid p_\Theta(z, Tz) - p_\Theta(z, Tz) \mid]$$

and so $p_\Theta(z, z) = 0$. □

Example 3.3. *Let* $X = [0, \frac{1}{2}]$. *Define the mappings* $\Theta : X \times X \to [1, \infty)$ *and* $d_\Theta : X \times X \to [0, \infty)$ *as follows:* $\Theta(x, y) = 1 + \frac{1}{1+x+y}$, $p_\Theta : X \times X \to [0, \infty)$ *such that* $p_\Theta(x, y) = y$ *and*

$$d_\Theta(x, y) = \begin{cases} x + y, & x, y \in X, x \neq y \\ 0, & x = y \end{cases}$$

Consider the mapping T *defined by* $T(x) = x^2$ *and partial order* $\preceq$ *given by*

$$x \preceq y \Leftrightarrow x = y \text{ or there exists } c \in [0, \frac{1}{2}] \text{ such that } y \leq \sqrt{\frac{c}{c+1}}x$$

It is clear that T is an increasing mapping. Moreover, there exists $x_0 = 0 \preceq Tx_0 = 0$. Now, we show that T is an extended P_{p_Θ}-contraction mapping for all $x \preceq y$. Now, we take into account the following two cases:

1) *If $x = y$, then $p_\Theta(Tx, Ty) = y^2 \leq \frac{1}{2}[y + 0] = \frac{1}{2}[p_\Theta(x, y)+ \mid p_\Theta(x, Tx) - p_\Theta(y, Ty) \mid]$. Thus, T is an extended P_{p_Θ}-contraction mapping with contraction constant $\kappa = \frac{1}{2}$.*

2) *If $x \neq y$, then there exists $c \in [0, \frac{1}{2}]$ such that $y \leq \sqrt{\frac{c}{c+1}}x$. Hence, $y^2 \leq \frac{c}{c+1}x^2$. Then, we obtain*

$$p_\Theta(Tx, Ty) = y^2 \tag{3.11}$$

and since $y \leq x$, we get $p_\Theta(x, y) + |p_\Theta(x, Tx) - p_\Theta(y, Ty)| = y + x^2 - y^2$. It can be clearly seen that

$$\begin{aligned} p_\Theta(Tx, Ty) = y^2 &\leq c[y + x^2 - y^2] \\ &= c[p_\Theta(x, y)+ \mid p_\Theta(x, Tx) - p_\Theta(y, Ty) \mid] \end{aligned}$$

Moreover, we have $\Theta(x, y) \leq 2 \leq \frac{1}{\kappa} = \frac{1}{c}$ for all $x, y \in X$ and T is continuous. It can be easily seen that conditions of the Theorem 3.3 are satisfied, so T has a fixed point.

Now we can give the following result as a corollary:

Corollary 3.1. *Let $(X, d_\Theta, \preceq)$ be a partially ordered complete extended b-metric space and p_Θ be an extended wt-distance on X and d_Θ continuous functional. Let $T : X \to X$ be an increasing mapping such that*

$$p_\Theta(Tx, Ty) \leq \kappa p_\Theta(x, y) \tag{3.12}$$

for all $x \preceq y$ or $y \in \Gamma((X, \tau_{d_\Theta}))$ and $\lim_{m,n\to\infty} \Theta(x_n, y_m) < \dfrac{1}{\kappa}$, where $x_n = T^n(x_0)$, for any $x_0 \in X$ and $\kappa \in [0, 1)$ and (y_m) be any sequence in X. If there exists a point $x_0 \in X$ with $x_0 \preceq Tx_0$ and one of the conditions listed below is true:

(i) T is continuous
(ii) p_Θ is continuous
(iii) for every $y \in X$ with $y \neq Ty$ and $x \preceq Tx$

$$inf\{p_\Theta(x, y) + p_\Theta(x, Tx) : x \in X\} > 0 \tag{3.13}$$

Then, T has a fixed point.

Now we extend our results to the cyclic case. To achieve this aim, we first introduce the following definition:

Definition 3.4. *Let $(X, d_\Theta, \preceq)$ be a partially ordered complete extended b-metric space and p_Θ be an extended wt-distance on X. Let t be a positive integer; $t \geq 2$, $A_1, A_2, ..., A_t$ be nonempty and closed subsets of X. Let $Y = \cup_{i=1}^t A_i$ and $T : Y \to Y$ be a mapping. Then, T is called cyclic extended P_{p_Θ}-contraction if it satisfies the followings:*

(i) $T(A_1) \subseteq A_2, ..., T(A_{t-1}) \subseteq A_t, T(A_t) \subseteq A_1$
(ii) for all $x \preceq y$ or $y \in \Gamma((X, \tau_{d_\Theta}))$ with $x \in A_i, y \in A_{i+1}$ such that

$$p_\Theta(Tx, Ty) \leq \kappa[p_\Theta(x, y)+ \mid p_\Theta(x, Tx) - p_\Theta(y, Ty) \mid] \tag{3.14}$$

and $\lim_{m,n\to\infty} \Theta(x_n, y_m) < \frac{1}{\kappa}$, where $x_n = T^n x_0, x_0 \in X$ and (y_m) be any sequence in X.

We give the cyclic version of our main theorem by the following result:

Theorem 3.4. *Let $(X, d_\Theta, \preceq)$ be a partially ordered complete extended b-metric space and p_Θ be an extended wt-distance on X and d_Θ continuous functional. Let $A_1, A_2, ..., A_t$ be nonempty and closed subsets of X and suppose that $T : \cup_{i=1}^{t} A_i \to \cup_{i=1}^{t} A_i$ be an increasing extended cyclic P_{p_Θ}-contraction that satisfies one of the conditions listed below:*

(i) T is continuous

(ii) p_Θ is continuous

(iii) for every $y \in \cup_{i=1}^{t} A_i$ with $y \neq Ty$ and $x \preceq Tx$

$$inf\{p_\Theta(x, y) + p_\Theta(x, Tx) : x \in X\} > 0 \tag{3.15}$$

If there exists a point $x_0 \in \cup_{i=1}^{t} A_i$ such that $x_0 \preceq Tx_0$, then T has a fixed point.

Proof. We will follow the similar steps with Theorem 3.3. Now, let $x_0 \in \cup_{i=1}^{t} A_i$ with $x_0 \preceq Tx_0$. Consider the sequence (x_n) defined by $x_{n+1} = Tx_n$ for $n \geq 0$. Due to the increasing mapping of T, we obtain by induction, $x_0 \preceq Tx_0 \preceq T^2 x_0 \preceq ... \preceq T^n x_0 \preceq T^{n+1} \preceq ...$ For any $n \in \mathbb{N}$, there is $i_n \in \{1, 2, ..., t\}$ such that $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_n+1}$. Now we consider the two cases:

Case I: Suppose that there exists $n_0 \in \mathbb{N}$ such that $p_\Theta(x_{n_0}, x_{n_0+1}) = 0$. Then, we have that $p_\Theta(x_{n_0+1}, x_{n_0+2}) = 0$. Indeed,

$$\begin{aligned} p_\Theta(x_{n_0+1}, x_{n_0+2}) &\leq \kappa[p_\Theta(x_{n_0}, x_{n_0+1}) + \mid p_\Theta(x_{n_0}, Tx_{n_0}) - p_\Theta(x_{n_0+1}, Tx_{n_0+1}) \mid] \\ &= \kappa[p_\Theta(x_{n_0}, x_{n_0+1}) + \mid p_\Theta(x_{n_0}, x_{n_0+1}) - p_\Theta(x_{n_0+1}, x_{n_0+2}) \mid] \\ &= \kappa p_\Theta(x_{n_0+1}, x_{n_0+2}) \end{aligned} \tag{3.16}$$

Since $\kappa < 1$, we have that $p_\Theta(x_{n_0+1}, x_{n_0+2}) = 0$ and by the condition (i) of extended wt-distance, we have the following inequality:

$$p_\Theta(x_{n_0}, x_{n_0+2}) \leq \Theta(x_{n_0}, x_{n_0+2})[p_\Theta(x_{n_0}, x_{n_0+1}) + p_\Theta(x_{n_0+1}, x_{n_0+2})] = 0 \tag{3.17}$$

Now we have that $p_\Theta(x_{n_0}, x_{n_0+1}) = 0$ and $p_\Theta(x_{n_0}, x_{n_0+2}) = 0$. By Lemma 2.1 we get that $x_{n_0+1} = x_{n_0+2} = Tx_{n_0+1}$. Then, x_{n_0+1} is a fixed point.

Case II: Now suppose that $p_\Theta(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$. Since $x_n \preceq x_{n+1}, x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_n+1}$. Then we have

$$\begin{aligned} p_\Theta(x_{n+1}, x_{n+2}) &= p_\Theta(Tx_n, Tx_{n+1}) \\ &\leq \kappa[p_\Theta(x_n, x_{n+1}) + \mid p_\Theta(x_n, Tx_n) - p_\Theta(x_{n+1}, Tx_{n+1}) \mid] \\ &= \kappa[p_\Theta(x_n, x_{n+1}) + \mid p_\Theta(x_n, x_{n+1}) - p_\Theta(x_{n+1}, x_{n+2}) \mid] \end{aligned} \tag{3.18}$$

This case requires it to be $p_\Theta(x_{n+1}, x_{n+2}) < p_\Theta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Hence from (3.18), we have that

$$p_\Theta(x_{n+1}, x_{n+2}) \leq \frac{2\kappa}{1+\kappa} p_\Theta(x_n, x_{n+1}) \tag{3.19}$$

for all $n \in \mathbb{N}$. Therefore we get $p_\Theta(x_{n+1}, x_{n+2}) \leq \lambda^{n+1} p_\Theta(x_0, x_1)$ for all $n \in \mathbb{N}$, where $\lambda = \dfrac{2\kappa}{\kappa + 1} < 1$.

Now we can show that (x_n) is a Cauchy sequence. For any $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} p_\Theta(x_n, x_m) &\leq \Theta(x_n, x_m)\lambda^n p_\Theta(x_0, x_1) + \Theta(x_n, x_m)\lambda^{n+1} p_\Theta(x_0, x_1) \\ &+ \Theta(x_n, x_m)\Theta(x_{n+1}, x_m)...\Theta(x_{m-1}, x_m)\lambda^{m-1} p_\Theta(x_0, x_1) \\ &\leq p_\Theta(x_0, x_1)[\Theta(x_1, x_m)\Theta(x_2, x_m)...\Theta(x_{n-1}, x_m)\lambda^n + \\ &+ \Theta(x_1, x_m)\Theta(x_2, x_m)...\Theta(x_n, x_m)\Theta(x_{n+1}, x_m)\lambda^{n+1} + ... \\ &+ \Theta(x_1, x_m)\Theta(x_2, x_m)...\Theta(x_n, x_m)\Theta(x_{n+1}, x_m)\Theta(x_{m-1}, x_m)\lambda^{m-1} \end{aligned}$$

Since $\lim_{m,n\to\infty} \Theta(x_{n+1}, x_m) < \dfrac{1}{\kappa}$, the series $\sum_{n=1}^{\infty} \lambda^n \prod_{r=1}^{n} \Theta(x_r, x_m)$ converges by ratio test for each $m \in \mathbb{N}$.

Let $\Lambda = \sum_{n=1}^{\infty} \lambda^n \prod_{r=1}^{n} \Theta(x_r, x_m)$ and $\Lambda_n = \sum_{n=1}^{n} \lambda^n \prod_{r=1}^{n} \Theta(x_r, x_m)$. Thus we have for $n < m$, $p_\Theta(x_n, x_m) \leq p_\Theta(x_0, x_1)[\Lambda_{m-1} - \Lambda_n]$, so from Lemma 2.1, we get that (x_n) is a Cauchy sequence in $\cup_{i=1}^{t} A_i$. Since $\cup_{i=1}^{t} A_i$ is a τ_{d_Θ}-closed, then $\cup_{i=1}^{t} A_i$ is complete. Therefore, there exists $z \in \cup_{i=1}^{t} A_i$ such that (x_n) is converging according to τ_{d_Θ}. It is observable that (x_n) has an infinite number of terms in each $A_i, i = 1, 2, ..., t$.

As (x_n) converges to $z \in \cup_{i=1}^{t} A_i$, it is possible to construct a subsequence of (x_n) that converges to z for each $A_i, i = 1, 2, ..., t$ and each each $A_i, i = 1, 2, ..., t$ is τ_{d_Θ}-closed gives us that $z \in A_i$ for each $i = 1, 2, ..., t$. Therefore, $z \in \bigcap_{i=1}^{t} A_i$.

Now we show that z is a fixed point of T. By using the Θ-lower semi-continuity of p_Θ, we have the following:

$$\begin{aligned} p_\Theta(x_n, z) &\leq \liminf_{m\to\infty} \Theta(x_n, z) p_\Theta(x_n, x_m) \\ &\leq \Theta(x_n, z) p_\Theta(x_0, x_1)[\Lambda_{m-1} - \Lambda_n] \end{aligned} \tag{3.20}$$

Now if T is continuous, then $x_{n+1} = Tx_n \to Tz$ and by the uniqueness of the limit we get that $z = Tz$. Moreover, $p_\Theta(z, z) = 0$. Indeed, since $z \preceq z$ and $z \in A_i$ for all $i = 1, 2, ..., t$ and by the contraction condition we have

$$p_\Theta(Tz, Tz) \leq \kappa[p_\Theta(z, z) + \mid p_\Theta(z, Tz) - p_\Theta(z, z) \mid]$$

Then, $p_\Theta(z, z) \leq \kappa p_\Theta(z, z)$ which is a contradiction unless $p_\Theta(z, z) = 0$. Now, if p_Θ is continuous, then we obtain

$$\lim_{n\to} p_\Theta(x_n, z) \leq \lim_{n\to\infty} \Theta(x_n, z) p_\Theta(x_0, x_1)[\Lambda_{m-1} - \Lambda_n] \tag{3.21}$$

and $\lim_{n\to\infty} p_\Theta(x_n, z) = 0$. Morever, $\lim_{m,n\to} p_\Theta(x_n, x_m) = p_\Theta(z, z) = 0$.

Now since z is an accumulation point of $\cup_{i=1}^{t} A_i$, $x_n \in A_{i_n}$ and $z \in A_{i_n+1}$ by putting $x = x_n$ and $y = z$ in the contraction condition we can write the following equation:

$$p_\Theta(x_{n+1}, Tz) \leq \kappa[p_\Theta(x_n, z) + \mid p_\Theta(x_n, x_{n+1}) - p_\Theta(z, Tz)] \tag{3.22}$$

for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ and using the continuity of p_Θ, we get that

$$p_\Theta(z, Tz) \leq c p_\Theta(z, Tz)$$

which is a contradiction. Thus, $p_\Theta(z, Tz) = 0$. Since we have $p_\Theta(z, z) = p_\Theta(z, Tz) = 0$ and from Lemma 2.1 we have $z = Tz$.

Finally, assume that condition (iii) holds and $z \neq Tz$. Then, for all $x \preceq Tx$ we have

$$0 < inf\{p_\Theta(x, z) + p_\Theta(x, Tx) : x \in \cup_{i=1}^{t} A_i\} \tag{3.23}$$

Since $x_n \preceq x_{n+1} = Tx_n$, we get

$$\begin{aligned} 0 &< inf\{p_\Theta(x_n, z) + p_\Theta(x_n, Tx_n) : x \in X\} \\ &= inf\{p_\Theta(x_n, z) + p_\Theta(x_n, x_{n+1}) : n \in \mathbb{N}\} \to 0, \text{ as } n \to \infty. \end{aligned}$$

which is a contradiction, thus $z = Tz$. Moreover since $z \preceq z$ by the contraction condition we have

$$p_\Theta(Tz, Tz) = p_\Theta(z, z) \leq \kappa[p_\Theta(z, z) + \mid p_\Theta(z, Tz) - p_\Theta(z, Tz) \mid]$$

and so $p_\Theta(z, z) = 0$. □

We end this section with the following example which shows the applicability of Theorem 3.4.

Example 3.4. *Let $X = \{1, 2, 3\}$ equipped with $d_\Theta : X \times X \to [0, \infty)$ given by $d_\Theta(x, y) = (x-y)^2$ and let us define $\Theta : X \times X \to [1, \infty)$ by $\Theta(x, y) = \frac{y+1}{2}$. It is clear that d_Θ is an extended b-metric on X. Consider the extended wt-distance $p_\Theta : X \times X \to [0, \infty)$ given by $p_\Theta(x, y) = y^2$. Let $A = \{1, 2\}$, $B = \{1, 3\}$ be nonempty closed subsets of X and $T : A \cup B \to A \cup B$ be a function defined by $T(1) = 1, T(2) = 3, T(3) = 2$. It is obvious that $T(A) = \{1, 3\} \subseteq \{B\}$ and $T(B) = \{1, 2\} \subseteq A$. Define a partial order $\preceq$ on X as follows:*

$$\preceq = \{(1, 1), (2, 2), (3, 3), (2, 1), (3, 1)\}$$

We can easily see that T is an increasing mapping according to partial order $\preceq$. Indeed, for $2 \preceq 1$ and $3 \preceq 1$, we have that $T2 = 3 \preceq T1 = 1$ and $T3 = 2 \preceq T1 = 1$. Morever, there exists $x_0 = 1 \in X$ such that $x_0 = 1 \preceq Tx_0 = 1$. Now we confirm the contraction condition for $2 \preceq 1$ such that $2 \in A$ and $1 \in B$:

$$p_\Theta(T2, T1) = p_\Theta(3, 1) = 1 \leq \kappa[p_\Theta(2, 1)+ \mid p_\Theta(2, T2) - p_\Theta(1, T1) \mid] = 9\kappa$$

and the contraction condition holds for $\kappa = \frac{1}{9}$. Morever, $\Theta(x, y) \leq 2 \leq \frac{1}{\kappa} = 9$. Therefore, all conditionsof Theorem 3.4 are satisfied and T has a fixed point.

4. Application

Now, in this part of the paper we verify the existence of the solution to the Volterra-Fredholm type integral equation which is the mathematical modelings of some important biological and physical models and given in [16] by the following:

$$u(t, x) = h(t, x) + \int_0^t \int_{\mathbb{R}^2} K(t, x, s, y, u(s, y))dyds, \forall(t, x) \in D \tag{4.24}$$

where $h : D \to \mathbb{R}^\mathbb{N}$, $K : D \times D \times \mathbb{R}^\mathbb{N} \to \mathbb{R}^\mathbb{N}$, $D = [0, T] \times \Omega, T > 0$ and Ω is the non empty and closed subset of Euclidean space $\mathbb{R}^\mathbb{N}$ is the non empty and closed set of Euclidean space $\mathbb{R}^\mathbb{N}$ equiped with norm $\|.\|$.

Let $(X, \|.\|)$ be a Banach space. Define the mapping $d_\Theta : X \times X \to [0, \infty)$ by

$$d_\Theta(x, y) = sup_{t\in D} \mid x(t) - y(t) \mid^2, \forall x, y \in X$$

It is easy to check that extended b-metric space (X, d_Θ) is complete where $\Theta : X \times X \to [1, \infty)$ with $\Theta(x, y) = 1 + \frac{|x(t)|+|y(t)|}{1+|x(t)|+|y(t)|}$.

Theorem 4.5. *Let $F : C^\mathbb{N}([a, b], X) \to C^\mathbb{N}([a, b], X)$ be selfmap of an extended b-metric space (X, d_Θ). Suppose the following assumptions hold:*

(1) the function $h : D \to \mathbb{R}^+$ and $K : D \times D \times \mathbb{R}^+ \to X$ are continuous;
(2) $K(t, x, .) : \mathbb{R}^\mathbb{N} \to \mathbb{R}^\mathbb{N}$ is increasing for each $t, x \in D$;
(3) there exists a continuous function $L : D \times D \to [0, \infty)$ such that

$$\mid K(t, x, s, y, u_1(t, x)) - K(t, x, s, y, u_2(t, x)) \mid \leq$$
$$[L(t, x, s, y) \mid (u_1(t, x) - Tu_1(t, x)) \mid^2 - \mid (u_2(t, x) - Tu_2(t, x)) \mid^2]^{\frac{1}{2}}$$

for all $t, x, s, y, u_1(t, x), u_2(t, x) \in D \times D \times \mathbb{R}^\mathbb{N}$ with $u_1 \preceq u_2$;
(4) There exists $u_0 \in C^\mathbb{N}([a, b], X)$ such that

$$u_0(t, x) \leq h(t, x) + \int_0^t \int_{\mathbb{R}^2} K(t, x, s, y, u_0(s, y))dyds$$

for any $t \in [a, b]$;

(5) *there is a path with a constant* $c \in [0,1)$ *and* $a, b \in [0,1]$ *such that*

$$\int_0^t \int_{\mathbb{R}^2} L(t,x,s,y)dyds \leq c$$

Then the Volterra-Fredholm integral equation has at least a solution in $C^{\mathbb{N}}([a,b],X)$.

Proof. In the proof we show all closed and bounded subsets of (X, d_Θ) by $P_{cl,b}(X)$. Let $A \in P_{cl,b}(X)$ and we define $p_\Theta : X \times X \to [0,\infty)$ where

$$p_\Theta(x,y) = \begin{cases} d_\Theta(x,y), & x,y \in A \\ \alpha, & x \notin A \text{ or } y \notin A \end{cases}$$

such that $\alpha \geq diamA$. It is easy to verify that p_Θ is an extended wt-distance. Consider the partial order $\preceq$ given by $x \preceq y \iff x(t) \leq y(t)$ for any $t \in [a,b]$. Let $F : C^{\mathbb{N}}([a,b],X) \to C^{\mathbb{N}}([a,b],X)$ be an operator such that $Fu(t,x) = h(t,x) + \int_0^t \int_{\mathbb{R}^2} K(t,x,s,y,u(s,y))dyds, \forall (t,x) \in D$. Due to the condition 2, F is increasing.

Case I: Let $x, y \in A$. For each $u_1, u_2 \in C^{\mathbb{N}}([a,b],X)$ with $u_1 \preceq u_2$, we have that

$$\begin{aligned}
(\mid Fu_1(t,x) - Fu_2(t,x) \mid)^2 &\leq \int_0^t \int_{\mathbb{R}^2} (\mid K(t,x,s,y,u_1(t,x)) - K(t,x,s,y,u_2(t,x)) \mid)^2 dyds \\
&\leq \int_0^t \int_{\mathbb{R}^2} L(t,x,s,y)(\mid (\mid u_1(t,x) - Fu_1(t,x) \mid)^2 \\
&- \mid u_2(t,x) - Fu_2(t,x) \mid)^2 \mid) dyds \\
&\leq (\mid (\mid u_1(t,x) - Fu_1(t,x) \mid)^2 - (\mid u_2(t,x) - Fu_2(t,x) \mid)^2 \mid) \\
&\times \int_0^t \int_{\mathbb{R}^2} L(t,x,s,y)dsdy \\
&\leq c(\mid (\mid u_1(t,x) - Fu_1(t,x) \mid)^2 - (\mid u_2(t,x) - Fu_2(t,x) \mid)^2 \mid) \\
&\leq c((\mid u_1(t,x) - u_2(t,x) \mid)^2 \mid (\mid u_1(t,x) - Fu_1(t,x) \mid)^2 \\
&- (\mid u_2(t,x) - Fu_2(t,x) \mid)^2 \mid)
\end{aligned}$$

Applying supremum on both sides we get that

$$\begin{aligned}
p_\theta(Fu_1(t,x), Fu_2(t,x)) &\leq c[p_\theta(u_1(t,x), u_2(t,x)) + \mid p_\theta(u_1(t,x), Fu_1(t,x)) \\
&- p_\theta(u_2(t,x), Fu_2(t,x)) \mid]
\end{aligned} \tag{4.25}$$

Case II: x or y does not belong to A. Then it is easy to remark that for this case the contraction condition with respect to p_θ is true.
From the condition 4 there exists u_0 with $u_0 \preceq Fu_0$. Further, $\lim_{m,n\to\infty}[1 + \frac{|x(t)|+|y(t)|}{1+|x(t)|+|y(t)|}] = 1 < \frac{1}{c}$. Then, from Theorem 3.3 we obtain that F has a fixed point. □

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DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND ARTS, ORDU UNIVERSITY, ORDU, TURKEY
Email address: `iremeroglu@odu.edu.tr`