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# Approximating fixed points of the SP*-iteration for generalized nonexpansive mappings in $CAT(0)$ spaces

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ABSTRACT. In this paper, In this paper we prove the strong and $\Delta-$convergence theorems of the SP*-iteration process for $C$-$\alpha$ nonexpansive mappings in $CAT(0)$ spaces. Moreover we study the data dependence result of the proposed iteration process for contraction mappings in $CAT(0)$ spaces. Also we provide an example that satisfies condition $C$-$\alpha$. Further we apply to the approximate the solution of the integral equation. Our results improve and extend some recently results in the literature of fixed point theory in $CAT(0)$ spaces.

## 1. INTRODUCTION AND PRELIMINARIES

A metric space $\mathcal{X}$ is a $CAT(0)$ space if it is geodesically connected and if every geodesic triangle in $\mathcal{X}$ is at least as 'thin' as its comparison triangle in the Euclidean plane. It is well-known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a $CAT(0)$ space. Other examples include Pre-Hilbert spaces, any convex subset of a Euclidian space $\mathbb{R}^n$ with the induced metric, the complex Hilbert ball with a hyperbolic metric and many others. For discussion of these spaces and of the fundamental role they play in geometry see Bridson and Haefliger [5]. Burago et al. [6] contains a somewhat more elementary treatment, and Gromov [13] a deeper study. Fixed point theory in $CAT(0)$ space has been first studied by Kirk (see [15],[16]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete $CAT(0)$ space always has a fixed point. On the other hand, we know that every Banach space is a $CAT(0)$ space. Since then the fixed point theory in CAT(0) has been rapidly developed and much papers a appeared.(see [8], [9], [12],[15]-[18]).

Recently, Kirk and Panyanak [18] used the concept of $\Delta-$convergence introduced by Lim [19] to prove on the $CAT(0)$ space analogs of some Banach space results which involve weak convergence. Further, Dhompongsa and Panyanak [8] obtained $\Delta-$convergence theorems for the Picard, Mann and Ishikawa iteration processes for nonexpansive mappings in the $CAT(0)$ space. If $x, y_1, y_2$ are points of a $CAT(0)$ spaces, and and if $y_0$ is the midpoint of the segment $[y_1, y_2]$ then the $CAT(0)$ inequality implies

$$d^2(x, y_0) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2).$$

This is the (CN) inequality of Bruhat and Tits [4]. In fact, a geodesic space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality ([[5], p. 163]).

In the sequel, we need the following definitions and useful lemmas to prove our main results of this paper.

**Lemma 1.1.** *([5], Proposition 2.2) Let $\mathcal{X}$ be a $CAT(0)$ space, $x, y, u, v \in \mathcal{X}$ and $t \in [0, 1]$. Then*

$$d(tx \oplus (1-t)y, tu \oplus (1-t)v) \leq td(x, u) + (1-t)d(y, v)$$

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**Lemma 1.2.** [8] *Let $\mathcal{X}$ be a $CAT(0)$ space.*

*(i) For $x, y \in \mathcal{X}$ and $t \in [0,1]$, there exists a unique point $z \in [x,y]$ such that $d(x,z) = td(x,y)$ and $d(y,z) = (1-t)d(x,y)$ (A). We use the notation $(1-t)x \oplus ty$ for the unique point $z$ satisfying (A).*

*(ii) For $x, y \in \mathcal{X}$ and $t \in [0,1]$, we have $d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z)$.*

**Lemma 1.3.** [8] *Let $\mathcal{X}$ be a $CAT(0)$ space. Then,*

$$d^2((1-t)x \oplus ty, z) \leq (1-t)d^2(x,z) + td^2(y,z) - t(1-t)d^2(x,y)$$

*for all $x, y, z \in \mathcal{X}$ and $t \in [0,1]$.*

**Proposition 1.1.** [26] *A geodesic space $\mathcal{X}$ is $CAT(0)$ if and only if for any $x, y, u, v \in \mathcal{X}$ we have*

$$d^2(x,u) + d^2(y,v) \leq d^2(x,y) + d^2(y,u) + d^2(u,v) + d^2(v,x). \tag{1.1}$$

First we present some basic concepts and definitions.

Let $\mathcal{X}$ be $CAT(0)$ space and $\mathcal{K}$ a nonempty subset of $\mathcal{X}$. Let $\mathcal{F} : \mathcal{K} \to \mathcal{K}$ be a mapping. A point $x \in \mathcal{K}$ is called a fixed point of $\mathcal{F}$ if $\mathcal{F}x = x$ and we denote by $Fix(\mathcal{F})$ the set of fixed points of $\mathcal{F}$, that is, $Fix(\mathcal{F}) = \{x \in \mathcal{K} : \mathcal{F}x = x\}$.

**Definition 1.1.** *Let $\{x_n\}$ be a bounded sequence in a closed convex subset $\mathcal{K}$ of a $CAT(0)$ space $\mathcal{X}$. For $x \in \mathcal{X}$, set $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$. The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by $r(\mathcal{K}, \{x_n\}) = \inf_n \{r(x, \{x_n\}) : x \in \mathcal{K}\}$ and the asymptotic center of $x_n$ relative to K is the set $A(\mathcal{K}, \{x_n\}) = \{x \in \mathcal{K} : r(x, \{x_n\}) = r(\mathcal{K}, \{x_n\})\}$. It is known that, in a $CAT(0)$ space, $A(\mathcal{K}, \{x_n\})$ consists of exactly one point; please, see* [11], *Proposition 7.*

*We now recall the definition of $\Delta$-convergence and weak convergence in $CAT(0)$ space.*

**Definition 1.2.** *(*[18],[19]*) A sequence $\{x_n\}$ in a $CAT(0)$ space $\mathcal{X}$ is said to $\Delta-$converge to $x \in \mathcal{X}$ if $\mathcal{X}$ is the unique asymptotic center of $u_n$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta - \lim_{n\to\infty} x_n = x$ and call $\mathcal{X}$ is the $\Delta-$limit of $\{x_n\}$.*

*A mapping $\mathcal{F} : \mathcal{K} \to \mathcal{K}$ is called contraction if there exists $\theta \in [0,1)$ such that*

$$d(\mathcal{F}x, \mathcal{F}y) \leq \theta d(x,y),$$

*for all $x, y \in \mathcal{K}$. If $\theta = 1$ in inequality above, then $\mathcal{F}$ is said to be a nonexpansive mapping.*

In 2008, Kirk and Panyanak [18] gave the following result for nonexpansive mappings on $CAT(0)$ spaces.

**Theorem 1.1.** [18] *Let $\mathcal{K}$ be a nonempty closed convex subset of a complete $CAT(0)$ space $\mathcal{X}$, and let $\mathcal{F} : \mathcal{K} \to \mathcal{K}$ be a nonexpansive mapping. Let $\{x_n\}$ be a bounded sequence in $\mathcal{K}$ with $\Delta - \limsup_{n\to\infty} x_n = x$ and $\limsup_{n\to\infty} d(x_n, \mathcal{F}x_n) = 0$. Then, $x \in \mathcal{K}$ and $\mathcal{F}x = x$.*

**Lemma 1.4.** *(*[18]*) Given $\{x_n\} \in \mathcal{X}$ such that $\{x_n\}$, $\Delta-$converges to $\mathcal{X}$ and given $y \in \mathcal{X}$ with $y \neq x$, then $\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y)$.*

**Lemma 1.5.** *(*[18]*) Every bounded sequence in a complete $CAT(0)$ space always has a $\Delta-$convergent subsequence.*

**Lemma 1.6.** *(*[10]*) Let $\mathcal{K}$ be closed convex subset of a complete $CAT(0)$ space and $\{x_n\}$ be a bounded sequence in $\mathcal{K}$. Then asymptotic center of $\{x_n\}$ is in $\mathcal{K}$.*

**Lemma 1.7.** [8] *Let $\mathcal{K}$ be a nonempty closed convex subset of $CAT(0)$ space $\mathcal{X}$. Let $\{x_n\}$ be a bounded sequence in $\mathcal{X}$ with $A(\{x_n\}) = \{x\}$, and let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$. Suppose that $\lim_{n\to\infty} d(x_n, u)$ exists. Then, $x = u$.*

In 2011, Lin et al.[20] gave the following some notations and lemmas.

Let $\{x_n\}$ be a bounded sequence in a $CAT(0)$ space $(\mathcal{X}, d)$, and let $\mathcal{K}$ be a nonempty closed convex subset of $\mathcal{X}$ which contains $\{x_n\}$. The notation $x_n \rightharpoonup w$ iff $\Psi(w) = inf_{x\in\mathcal{K}}\Psi(x)$, where $\Psi(x) := \limsup_{n\to\infty} d(x_n, x)$. Then, we observe that

$$A(\{x_n\}) = x \in \mathcal{X} : \Psi(x) = inf_{u\in\mathcal{X}}\Psi(u)$$

and

$$A_{\mathcal{K}}(\{x_n\}) = x \in \mathcal{K} : \Psi(x) = inf_{u\in\mathcal{K}}\Psi(u).$$

**Lemma 1.8.** [20] *Let $\{x_n\}$ be a bounded sequence in a $CAT(0)$ space $\mathcal{X}$, and let $\mathcal{K}$ be a nonempty closed convex subset of $\mathcal{X}$ which contains $\{x_n\}$. If $x_n \rightharpoonup w$, then $w \in \mathcal{K}$.*

**Lemma 1.9.** [21] *Let $\mathcal{K}$ be a nonempty closed convex subset of $CAT(0)$ space $\mathcal{X}$, and let $\{x_n\}$ be a bounded sequence in $\mathcal{K}$. If $\Delta - lim_n x_n = x$, then $x_n \rightharpoonup x$.*

A number of extensions and generalizations of nonexpansive mappings have been considered by many mathematicians, see [[1], [22], [27]], in recent years. In 2008, Suzuki [27] introduced the concept of generalized nonexpansive mappings which is a condition on mappings called $(C)$ *condition*. Let $\mathcal{K}$ be a nonempty convex subset of a Banach space $\mathcal{X}$, a mapping $\mathcal{F} : \mathcal{K} \to \mathcal{K}$ is satisfy $(C)$ *condition* if for all $x, y \in \mathcal{K}$, $\frac{1}{2}d(x, \mathcal{F}x) \leq d(x, y)$ implies $d(\mathcal{F}x, \mathcal{F}y) \leq d(x, y)$. Suzuki [27] showed that the mapping satisfying $(C)$ *condition* is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness. In 2011, Aoyama and Kohsaka [1] introduced the class of $\alpha$-nonexpansive mappings in the setting of Banach spaces and obtained some fixed point results for such mappings. In 2017, Pant and Shukla [22] introduced the following class of nonexpansive type mappings and obtained some fixed point results for this class of mappings.This class of nonlinear mappings properly contains nonexpansive, Suzuki-type generalized nonexpansive mappings and partially extends firmly nonexpansive and $\alpha$-nonexpansive mappings.

In what follows, we give the following definition and lemma to be used in main results.

**Definition 1.3.** [22]. *A mapping $\mathcal{F} : \mathcal{K} \to \mathcal{K}$ is called a generalized $\alpha$-nonexpansive mapping if there exists an $\alpha \in [0, 1)$ and for each $x, y \in \mathcal{K}$,*

$$\frac{1}{2}d(x, \mathcal{F}x) \leq d(x, y) \text{ implies } d(\mathcal{F}x, \mathcal{F}y) \leq \alpha d(\mathcal{F}x, y) + \alpha d(\mathcal{F}y, x) + (1 - 2\alpha)d(x, y).$$

Furthermore, in [23], authors presented the following new class of nonexpansive type mappings and obtained some fixed point results for this new class of mappings.

**Definition 1.4.** [23] *A mapping $\mathcal{F} : \mathcal{K} \to \mathcal{K}$ is called $C$-$\alpha$ nonexpansive mapping if there exists an $\alpha \in [0, 1)$ and for each $x, y \in \mathcal{K}$,*

$$\begin{aligned}&\frac{1}{2}d(x, \mathcal{F}x) \leq d(x, y) \text{ implies}\\ &d^2(\mathcal{F}x, \mathcal{F}y) \leq \alpha d^2(\mathcal{F}x, y) + \alpha d^2(x, \mathcal{F}y) + (1 - 2\alpha)d^2(x - y).\end{aligned}$$

A mapping satisfying the condition $(C)$ is $C$-$\alpha$ nonexpansive mapping. An $\alpha-$nonexpansive mapping is a $C$-$\alpha$ nonexpansive mapping and also generalized $\alpha-$nonexpansive mapping is a $C$-$\alpha$ nonexpansive mapping, but from the examples given in [23] and [30] it can be seen that the reverse is not true.

**Proposition 1.2.** *Let $\mathcal{K}$ be a nonempty closed convex subset of a complete $CAT(0)$ space $\mathcal{X}$ and let $\mathcal{F} : \mathcal{K} \to \mathcal{X}$ be a $C$-$\alpha$ nonexpansive mapping with $Fix(\mathcal{F}) \neq \emptyset$. Then, $Fix(\mathcal{F})$ is a closed convex subset of $\mathcal{K}$.*

*Proof.* If $x_n$ is a sequence in $Fix(\mathcal{F})$ and $\lim_{n\to\infty} x_n = x$. Then, we have:

$$d^2(\mathcal{F}x, x_n) \leq d^2(x_n, x) + \frac{\alpha}{1-\alpha} d^2(\mathcal{F}x, x). \tag{1.2}$$

This implies that

$$\frac{1-2\alpha}{1-\alpha} d^2(\mathcal{F}x, x) \leq 0.$$

Then $\mathcal{F}x = x$ and $Fix(\mathcal{F})$ is a closed set. Next, we want to show that $Fix(\mathcal{F})$ is a convex set. If $x, y \in Fix(\mathcal{F}) \subseteq K$ and $z \in [x, y]$, then there exists $t \in [0,1]$ such that $z = tx \oplus (1-t)y$. Since $\mathcal{K}$ is convex, $z \in \mathcal{K}$. Furthermore,

$$\begin{aligned} d^2(\mathcal{F}z, z) &\leq td^2(\mathcal{F}z, x) + (1-t)d^2(\mathcal{F}z, y) - t(1-t)d^2(x, y) \\ &\leq td^2(z, x) + \frac{t\alpha}{1-\alpha} d^2(\mathcal{F}z, z) + (1-t)d^2(z, y) \\ &\quad + \frac{(1-t)\alpha}{1-\alpha} d^2(\mathcal{F}z, z) - t(1-t)d^2(x, y) \\ &= t(1-t)^2 d^2(y, x) + \frac{\alpha}{1-\alpha} d^2(\mathcal{F}z, z) \\ &\quad + t^2(1-t)d^2(x, y) - t(1-t)d^2(x, y) \\ &\leq \frac{\alpha}{1-\alpha} d^2(\mathcal{F}z, z) \end{aligned}$$

Hence, $\mathcal{F}z = z$ and $Fix(F)$ is a convex set. □

The concept of approximating fixed points for generalized nonexpansive mappings plays an important role in the study of three-step iteration processes. Pant and Shukla [23] studied the Noor iteration scheme for $C$-$\alpha$ nonexpansive mapping.

Let $\mathcal{X}$ be a real Banach space and $\mathcal{K}$ be a nonempty subset of $\mathcal{X}$, and $\mathcal{F} : \mathcal{K} \to \mathcal{K}$ be a mapping. We have $\{a_n\}, \{b_n\}$ and $\{c_n\}$ real sequences in $[0,1]$. Recently, Phuengrattana and Suantai ([24]) defined the SP-iteration as follows:

$$\begin{cases} z_n = (1-c_n)x_n + c_n \mathcal{F}x_n, \\ y_n = (1-b_n)z_n + b_n \mathcal{F}z_n, \\ x_{n+1} = (1-a_n)y_n + a_n \mathcal{F}y_n, \forall n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $x_1 \in \mathcal{K}$. They showed that the Mann, Ishikawa, Noor and SP-iteration are equivalent and the SP-iteration converges better than the others for the class of continuous and nondecreasing functions. In 2014, Basarır and Şahin [3] studied S-iteration process for generalized nonexpansive mappings on $CAT(0)$ space. In 2014, Kadıoglu and Yıldırım [14] introduced Picard Normal S-iteration process and they established that the rate of convergence of the Picard Normal S-iteration process is faster than other fixed point iteration process that was in existence then. The Picard Normal S-iteration [14] as follows:

$$\begin{cases} z_n = (1-b_n)x_n + b_n \mathcal{F}x_n, \\ y_n = (1-a_n)z_n + a_n \mathcal{F}z_n, \\ x_{n+1} = \mathcal{F}y_n, \forall n \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $x_1 \in \mathcal{K}$.

In 2021, Temir and Korkut [31] introduced SP*-iteration scheme and they established that the rate of convergence of the SP*-iteration scheme is faster than above fixed point iteration process. Now we give SP*-iteration scheme:for arbitrary $x_1 \in \mathcal{K}$ construct a sequence $\{x_n\}$ by

$$\text{(1.5)} \qquad \begin{cases} z_n = \mathcal{F}((1-c_n)x_n + c_n\mathcal{F}x_n), \\ y_n = \mathcal{F}((1-b_n)z_n + b_n\mathcal{F}z_n), \\ x_{n+1} = \mathcal{F}((1-a_n)y_n + a_n\mathcal{F}y_n), \forall n \in \mathbb{N}, \end{cases}$$

In this paper, we apply SP*-iteration (1.5) in setting of $CAT(0)$ space for generalized nonexpansive mappings as follows

$$\text{(1.6)} \qquad \begin{cases} z_n = \mathcal{F}((1-c_n)x_n \oplus c_n\mathcal{F}x_n), \\ y_n = \mathcal{F}((1-b_n)z_n \oplus b_n\mathcal{F}z_n), \\ x_{n+1} = \mathcal{F}((1-a_n)y_n \oplus a_n\mathcal{F}y_n) \forall n \in \mathbb{N}, \end{cases}$$

where $\mathcal{K}$ is a nonempty closed convex subset of a $CAT(0)$ space, $x_1 \in \mathcal{K}$, $\{a_n\}$, $\{b_n\}$ and $\{c_n\} \in [0,1]$.

In this paper, we study the convergence and the data dependence of the iteration process (1.6) in $CAT(0)$ spaces. This paper contains six sections. In Section 2, we give some fixed point theorems and demi-closed principle of $C$-$\alpha$ nonexpansive mapping on $CAT(0)$ spaces. In Section 3, we prove some results related to the strong and $\Delta-$convergence of SP*-iteration process (1.6) for $C$-$\alpha$ nonexpansive mapping. In Section 4, also we give an illustrative numerical example that satisfies $C$-$\alpha$ nonexpansive mapping. In Section 5, we prove the data dependence result of the SP*-iteration process (1.6) for contraction mappings in $CAT(0)$ spaces. In Section 6, we give application of SP*-iteration process for integral equation.

## 2. Fixed point theorems on complete $CAT(0)$ spaces

The following theorem establishes a demiclosed principle for a $C$-$\alpha$ nonexpansive mapping on $CAT(0)$ spaces.

**Theorem 2.2.** *Let $\mathcal{K}$ be a nonempty closed convex subset of a complete $CAT(0)$ space $\mathcal{X}$, and let $\mathcal{F} : \mathcal{K} \to \mathcal{X}$ be a $C$-$\alpha$ nonexpansive mapping. Let $\{x_n\}$ be a bounded sequence in $\mathcal{K}$ with $x_n \rightharpoonup x$ and $\lim_{n\to\infty} d(x_n, \mathcal{F}x_n) = 0$. Then, $x \in \mathcal{K}$ and $\mathcal{F}x = x$.*

*Proof.* Since $x_n \rightharpoonup x$, we know that $x \in \mathcal{K}$ and $\Psi(x) = inf_{u\in\mathcal{K}}\Psi(u)$, where $\Psi(u) := \limsup_{n\to\infty} d(x_n, u)$. Furthermore, we know that $\Psi(x) = inf \Psi(u) : u \in \mathcal{X}$. Since $\mathcal{F}$ is a $C$-$\alpha$ nonexpansive mapping,

$$d^2(\mathcal{F}x_n, \mathcal{F}x) \le \alpha d^2(\mathcal{F}x_n, x) + \alpha d^2(x_n, \mathcal{F}x) + (1-2\alpha)d^2(x_n, x)$$

By (1.1), then we have

$$\begin{aligned} d^2(\mathcal{F}x_n, \mathcal{F}x) \quad &\le \quad \alpha d^2(\mathcal{F}x_n, x_n) + \alpha d^2(x_n, x) + \alpha d^2(\mathcal{F}x, x) \\ &\quad + \alpha d^2(\mathcal{F}x_n, \mathcal{F}x) + (1-2\alpha)d^2(x_n, x) \end{aligned}$$

Thus we have

$$(1-\alpha)d^2(\mathcal{F}x_n, \mathcal{F}x) \le \alpha d^2(\mathcal{F}x, x) + (1-\alpha)d^2(x_n, x)$$

Hence we have

$$\limsup_{n\to\infty} d^2(\mathcal{F}x_n, \mathcal{F}x) \le \limsup_{n\to\infty} d^2(x_n, x) + \frac{\alpha}{(1-\alpha)} d^2(\mathcal{F}x, x).$$

This implies that

$$\begin{aligned}\limsup_{n\to\infty} d^2(x_n,\mathcal{F}x) &\le \limsup_{n\to\infty}(d(x_n,\mathcal{F}x_n)+d(\mathcal{F}x_n,\mathcal{F}x))^2\\ &\le \limsup_{n\to\infty} d^2(\mathcal{F}x_n,\mathcal{F}x)\\ &\le \limsup_{n\to\infty} d^2(x_n,x)+\frac{\alpha}{(1-\alpha)}d^2(\mathcal{F}x,x).\end{aligned}$$

Besides , by (CN) inequality, we have

$$d^2(x_n,\frac{1}{2}x\oplus\frac{1}{2}\mathcal{F}x)\le\frac{1}{2}d^2(x_n,x)+\frac{1}{2}d^2(x_n,\mathcal{F}x)-\frac{1}{4}d^2(x,\mathcal{F}x).$$

So, take limsup both sides, then we have

$$\begin{aligned}\limsup_{n\to\infty} d^2(x_n,\frac{1}{2}x\oplus\frac{1}{2}\mathcal{F}x) &\le \frac{1}{2}\limsup_{n\to\infty} d^2(x_n,x)+\frac{1}{2}\limsup_{n\to\infty} d^2(x_n,x)\\ &\quad -\frac{1}{4}\limsup_{n\to\infty} d^2(\mathcal{F}x,x)\\ &\le \limsup_{n\to\infty} d^2(x_n,x)+\frac{\alpha}{2(1-\alpha)}d^2(x,\mathcal{F}x)-\frac{1}{4}d^2(\mathcal{F}x,x).\end{aligned}$$

So, we have

$$(\frac{1}{4}-\frac{\alpha}{2(1-\alpha)})d^2(\mathcal{F}x,x)\le\limsup_{n\to\infty} d^2(x_n,x)-\limsup_{n\to\infty} d^2(x_n,\frac{1}{2}x\oplus\frac{1}{2}\mathcal{F}x).$$

Hence

$$(\frac{1}{4}-\frac{\alpha}{2(1-\alpha)})d^2(\mathcal{F}x,x)\le\Psi(x)^2-(\Psi(\frac{1}{2}x\oplus\frac{1}{2}\mathcal{F}x))^2\le 0.$$

Therefore, $\mathcal{F}x=x$. □

Thus it is easy to get the following result for $C$-$\alpha$ nonexpansive mapping.

**Theorem 2.3.** *Let $\mathcal{K}$ be a nonempty closed convex subset of a complete $CAT(0)$ space $\mathcal{X}$, and let $\mathcal{F}:\mathcal{K}\to\mathcal{X}$ be a $C$-$\alpha$ nonexpansive mapping. Let $\{x_n\}$ be a bounded sequence in $\mathcal{K}$ with $\Delta-\limsup_{n\to\infty} x_n=x$ and $\limsup_{n\to\infty} d(x_n,\mathcal{F}x_n)=0$. Then, $x\in\mathcal{K}$ and $\mathcal{F}x=x$.*

**Theorem 2.4.** *Let $\mathcal{K}$ be a nonempty closed convex subset of a complete $CAT(0)$ space $\mathcal{X}$, and let $\mathcal{F}:\mathcal{K}\to\mathcal{K}$ be a $C$-$\alpha$ nonexpansive mapping for all $x\in\mathcal{K}$. Then, the following conditions are equivalent:*
*(i) $\{\mathcal{F}^n x\}$ is bounded for some $x\in\mathcal{K}$;*
*(ii) $Fix(\mathcal{F})\ne\emptyset$.*

*Proof.* Suppose that $\{\mathcal{F}^n x\}$ is bounded for some $x\in\mathcal{K}$. For each $n\in\mathbb{N}$, let $x_n=\mathcal{F}^n x$. Since $\{x_n\}$ is bounded, there exists $\bar{x}\in\mathcal{X}$ such that $A(\{x_n\})=\{\bar{x}\}$. By Lemma 1.6, $\bar{x}\in\mathcal{K}$. Furthermore, we have

$$d^2(x_n,\mathcal{F}\bar{x})\le\alpha d^2(\bar{x},x_n)+\alpha d^2(x_{n-1},\mathcal{F}\bar{x})+(1-2\alpha)d^2(x_{n-1},\bar{x})$$

This implies that

$$\begin{aligned}\limsup_{n\to\infty} d^2(x_n,\mathcal{F}\bar{x}) &\le \alpha\limsup_{n\to\infty} d^2(\bar{x},x_n)+\alpha\limsup_{n\to\infty} d^2(x_{n-1},\mathcal{F}\bar{x})\\ &\quad +(1-2\alpha)\limsup_{n\to\infty} d^2(x_{n-1},\bar{x})\\ &\le (1-\alpha)\limsup_{n\to\infty} d^2(\bar{x},x_n)+\alpha\limsup_{n\to\infty} d^2(x_n,\mathcal{F}\bar{x})\end{aligned}$$

So we have

$$(1-\alpha)\limsup_{n\to\infty} d^2(x_n,\mathcal{F}\bar{x}) \leq (1-\alpha)\limsup_{n\to\infty} d^2(\bar{x},x_n)$$

Finally,

$$\begin{aligned}(\Psi(\mathcal{F}\bar{x}))^2 &= \limsup_{n\to\infty} d^2(x_n,\mathcal{F}\bar{x})\\ &\leq \limsup_{n\to\infty} d^2(\bar{x},x_n) = (\Psi(\bar{x}))^2.\end{aligned}$$

Since $A(\{x_n\}) = \{\bar{x}\}$, $\mathcal{F}\bar{x} = \bar{x}$. This shows that $Fix(\mathcal{F}) \neq \emptyset$. □

Next, we prove some convergence theorems of SP*-iteration process generated by (1.6) to fixed point for $C$-$\alpha$ nonexpansive mappings in $CAT(0)$ spaces.

## 3. Convergence of SP*-iteration process for $C$-$\alpha$ nonexpansive mappings

**Lemma 3.10.** *Let $\mathcal{K}$ be a nonempty closed convex subset of a complete $CAT(0)$ space $\mathcal{X}$, $\mathcal{F}$ be a $C$-$\alpha$ nonexpansive mapping with $Fix(\mathcal{F}) \neq \emptyset$. For arbitrary chosen $x_1 \in \mathcal{K}$, let $\{x_n\}$ be a sequence generated by* (1.6) *with $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ real sequences in $[0,1]$. Assume that $\liminf_{n\to\infty}(1-c_n)c_n > 0, \liminf_{n\to\infty}(1-b_n)b_n > 0$ and $\liminf_{n\to\infty}(1-a_n)a_n > 0$. Then $Fix(\mathcal{F}) \neq \emptyset$ if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n,\mathcal{F}x_n) = 0$.*

*Proof.* For any $p \in Fix(\mathcal{F})$, and $x \in \mathcal{K}$, since for$\mathcal{F}$ a $C$-$\alpha$ nonexpansive mapping, $\frac{1}{2}d(p,\mathcal{F}p) = 0 \leq d(p,x)$ implies that

$$\begin{aligned}d^2(\mathcal{F}p,\mathcal{F}x) &\leq \alpha d^2(\mathcal{F}p,x) + \alpha d^2(\mathcal{F}x,p) + (1-2\alpha)d^2(p,x)\\ &\leq \alpha d^2(\mathcal{F}p,x) + \alpha d^2(\mathcal{F}p,\mathcal{F}x) + (1-2\alpha)d^2(p,x)\\ (1-\alpha)d^2(\mathcal{F}p,\mathcal{F}x) &\leq \alpha d^2(\mathcal{F}p,x) + (1-2\alpha)d^2(p,x)\\ &= (1-\alpha)d^2(p,x).\end{aligned}$$

Thus, $d(\mathcal{F}p,\mathcal{F}x) \leq d(p,x)$ for all $x \in \mathcal{K}$. Now, using (1.6), we have

$$\begin{aligned}d^2(z_n,p) &= d^2(\mathcal{F}((1-c_n)x_n \oplus c_n\mathcal{F}x_n),p)\\ &\leq d^2((1-c_n)x_n \oplus c_n\mathcal{F}x_n,p)\\ &\leq (1-c_n)d^2(x_n,p) + c_nd^2(\mathcal{F}x_n,p) - (1-c_n)c_nd^2(\mathcal{F}x_n,x_n)\\ &\leq d^2(x_n,p) - (1-c_n)c_nd^2(\mathcal{F}x_n,x_n)\\ &\leq d^2(x_n,p).\end{aligned} \tag{3.7}$$

Using (1.6) and (3.7), we get

$$\begin{aligned}d^2(y_n,p) &= d^2(\mathcal{F}((1-b_n)z_n \oplus b_n\mathcal{F}z_n),p)\\ &\leq d^2((1-b_n)z_n \oplus b_n\mathcal{F}z_n,p)\\ &\leq (1-b_n)d^2(z_n,p) + b_nd^2(\mathcal{F}z_n,p) - (1-b_n)b_nd^2(\mathcal{F}z_n,z_n)\\ &\leq d^2(z_n,p) - (1-b_n)b_nd^2(\mathcal{F}z_n,z_n)\\ &\leq d^2(z_n,p) \leq d^2(x_n,p).\end{aligned} \tag{3.8}$$

By using (1.6) and (3.8), we get

$$\begin{aligned}
(3.9)\qquad d^2(x_{n+1},p) &= d^2(\mathcal{F}((1-a_n)y_n \oplus a_n\mathcal{F}y_n),p)\\
&\leq d^2((1-a_n)y_n \oplus a_n\mathcal{F}y_n,p)\\
&\leq (1-a_n)d^2(y_n,p)+a_nd^2(\mathcal{F}y_n,p)-(1-a_n)a_nd^2(\mathcal{F}y_n,y_n)\\
&\leq (1-a_n)d^2(x_n,p)+a_nd^2(y_n,p)-(1-a_n)a_nd^2(\mathcal{F}y_n,y_n)\\
&\leq (1-a_n)d^2(x_n,p)+a_nd^2(x_n,p)-(1-a_n)a_nd^2(\mathcal{F}y_n,y_n)\\
&\leq d^2(x_n,p)-(1-a_n)a_nd^2(\mathcal{F}y_n,y_n)\\
&\leq d^2(x_n,p).
\end{aligned}$$

This implies that $\{d(x_n,p)\}$ is bounded and non-increasing for all $p \in Fix(\mathcal{F})$. Put $\lim\limits_{n\to\infty} d(x_n,p)=c$. From (3.7) and (3.8), we have

$$\limsup_{n\to\infty} d(z_n,p) \leq \limsup_{n\to\infty} d(x_n,p)=c$$

and

$$\limsup_{n\to\infty} d(y_n,p) \leq \limsup_{n\to\infty} d(z_n,p) \leq \limsup_{n\to\infty} d(x_n,p)=c.$$

On the other hand,

$$\begin{aligned}
d(x_{n+1},p) &= d(\mathcal{F}((1-a_n)y_n \oplus a_n\mathcal{F}y_n,p))\\
&\leq (1-a_n)d(y_n,p)+a_nd(\mathcal{F}y_n,p)\\
&\leq (1-a_n)d(y_n,p)+a_nd(y_n,p)\\
&\leq d(y_n,p).
\end{aligned}$$

So we can get $d(x_{n+1},p) \leq d(y_n,p)$. Therefore $c \leq \liminf\limits_{n\to\infty} d(y_n,p)$. Thus we have $c = \lim\limits_{n\to\infty} d(y_n,p)$. Next

$$c=\lim_{n\to\infty} d(y_n,p) \leq \lim_{n\to\infty} d(z_n,p) \leq \lim_{n\to\infty} d(x_n,p)=c.$$

Now, using (3.7), we know that

$$d^2(z_n,p) \leq d^2(x_n,p)-(1-c_n)c_nd^2(\mathcal{F}x_n,x_n).$$

Thus

$$(1-c_n)c_nd^2(\mathcal{F}x_n,x_n) \leq d^2(x_n,p)-d^2(z_n,p)$$

so that

$$d^2(\mathcal{F}x_n,x_n) \leq \frac{1}{(1-c_n)c_n}[d^2(x_n,p)-d^2(z_n,p)]$$

We have

$$\lim_{n\to\infty} d^2(\mathcal{F}x_n,x_n) \leq 0.$$

Hence $\lim\limits_{n\to\infty} d(\mathcal{F}x_n,x_n)=0$.

Conversely, suppose that $\{x_n\}$ is bounded $\lim\limits_{n\to\infty} d(x_n,\mathcal{F}x_n)=0$. Let $p \in A(\mathcal{K},\{x_n\})$. So, we know that

$$\limsup_{n\to\infty} d^2(x_n,\mathcal{F}p) \leq \limsup_{n\to\infty} d^2(p,x_n).$$

Finally,

$$\begin{aligned}
(\Psi(\mathcal{F}p))^2 &= \limsup_{n\to\infty} d^2(x_n,\mathcal{F}p)\\
&\leq \limsup_{n\to\infty} d^2(p,x_n)=(\Psi(p))^2.
\end{aligned}$$

Since $A(\{x_n\}) = \{p\}$, $\mathcal{F}p = p$. This shows that $Fix(\mathcal{F}) \neq \emptyset$. This implies that for $\mathcal{F}p = p \in A(\mathcal{K}, \{x_n\})$. Since $\mathcal{X}$ is complete $CAT(0)$ then $A(\mathcal{K}, \{x_n\})$ is singleton. This completes the proof. □

Now , we prove the $\Delta-$convergence theorem of a iterative process generated by (1.6) in $CAT(0)$ spaces.

**Theorem 3.5.** *Let $\mathcal{X}, \mathcal{K}, \mathcal{F}$ and $\{x_n\}$ be as in Lemma 3.10 with $Fix(\mathcal{F}) \neq \emptyset$. Then $x_n$, $\Delta-$converges to a fixed point of $\mathcal{F}$.*

*Proof.* Lemma 3.10 guarantees that the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(\mathcal{F}x_n, x_n) = 0$. Let $W_\Delta(x_n) = \bigcup A(\{u_n\})$; where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$ : We claim that $W_\Delta(x_n) \subseteq Fix(\mathcal{F})$. Let $u \in W_\Delta(x_n)$. Then, there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = u$. By Lemma 1.5 and Lemma 1.6, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_{n\to\infty} v_n = v \in \mathcal{K}$. Now, we claim that $u = v$. Assume on contrary, that $u \neq v$. By Lemma 3.10, $\lim_{n\to\infty} d(x_n, v)$ exists and by the uniqueness of asymptotic centers, then we have

$$\begin{aligned}
\lim_{n\to\infty} d(v_n, v) &< \lim_{n\to\infty} d(v_n, u) \leq \lim_{n\to\infty} d(u_n, u) \\
&< \lim_{n\to\infty} d(u_n, v) = \lim_{n\to\infty} d(x_n, v) \\
&= \lim_{n\to\infty} d(v_n, v),
\end{aligned}$$

which is contraction. Thus $u = v \in Fix(\mathcal{F})$ and $W_\Delta(x_n) \subseteq Fix(\mathcal{F})$. To show that $\{x_n\}$, $\Delta$converges to a fixed point of$\mathcal{F}$, we show that $W(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$. By Lemma 1.5 and Lemma 1.6, there exists a subsequence $\{v_n\}$ of $u_n$ such that $\Delta - \lim_{n\to\infty} v_n = v \in \mathcal{K}$. Let $A(\{u_n\}) = \{u\}$ and $A(\{x_n\}) = \{x\}$. We have already seen that $u = v$ and $v \in Fix(\mathcal{F})$. Finally, we claim that $x = v$. If not, then existence $\lim_{n\to\infty} d(x_n, v)$ and uniqueness of asymptotic centers imply that

$$\begin{aligned}
\lim_{n\to\infty} d(v_n, v) &< \lim_{n\to\infty} d(v_n, x) \leq \lim_{n\to\infty} d(x_n, x) \\
&< \lim_{n\to\infty} d(x_n, v) = \lim_{n\to\infty} d(v_n, v).
\end{aligned}$$

This is a contradiction and hence $x = v \in Fix(\mathcal{F})$. Therefore, $W_\Delta(x_n) = x$. □

In the next result, we prove the strong convergence theorem as follows.

**Theorem 3.6.** *Let $\mathcal{F}$ be a $C$-$\alpha$ nonexpansive mapping on a compact convex subset $\mathcal{K}$ of a complete $CAT(0)$ space $\mathcal{X}$. $\{x_n\}$ be as in Lemma 3.10 with $Fix(\mathcal{F}) \neq \emptyset$. Then $\{x_n\}$ converges strongly to a fixed point of $\mathcal{F}$.*

*Proof.* By Lemma 3.10, we have $\lim_{n\to\infty} d(x_n, \mathcal{F}x_n) = 0$. Since $\mathcal{K}$ is compact, by Lemma 1.5, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $p \in \mathcal{K}$ such that $\{x_{n_k}\}$ converges $p$. By (1.2), we have $d(x_{n_k}, \mathcal{F}p) \leq \frac{\alpha}{1-\alpha} d(\mathcal{F}x_{n_k}, x_{n_k}) + d(x_{n_k}, p)$ for all $k \geq 0$. Then $\{x_{n_k}\}$ converges $\mathcal{F}p$. This implies $\mathcal{F}p = p$. Since$\mathcal{F}$ is quasinonexpansive , we have $d(x_{n+1}, p) \leq d(x_n, p)$ for all $n \in \mathbb{N}$. Therefore $\{x_n\}$ converges strongly to $p$. □

Finally, we briefly discuss the strong convergence theorem using condition $(I)$ introduced by Senter and Dotson[25] in $CAT(0)$ space $\mathcal{X}$ as follows.

**Theorem 3.7.** *Let $\mathcal{X}, \mathcal{K}, \mathcal{F}$ and $\{x_n\}$ be as in Lemma 3.10 with $Fix(\mathcal{F}) \neq \emptyset$. Also if, for$\mathcal{F}$ satisfies condition $(I)$, then $\{x_n\}$ defined by* (1.6) *converges strongly to a fixed point of $\mathcal{F}$.*

*Proof.* By Lemma 3.10, we have $\lim_{n\to\infty} d(x_n, p)$ exists and so $\lim_{n\to\infty} d(x_n, Fix(\mathcal{F}))$ . Also by Lemma 3.10, $\lim_{n\to\infty} d(x_n, \mathcal{F}x_n) = 0$.

It follows from condition $(I)$ that $\lim_{n\to\infty} f(d(x_n, Fix(\mathcal{F})) \leq \lim_{n\to\infty} d(x_n, \mathcal{F}x_n)$. That is, $\lim_{n\to\infty} f(d(x_n, Fix(\mathcal{F})) = 0$. Since $f : [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$, we have $\lim_{n\to\infty} d(x_n, Fix(\mathcal{F})) = 0$. Thus , we have a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $\{y_k\} \subset Fix(\mathcal{F})$ such that $d(x_{n_k}, y_k) < \frac{1}{2^k}$ for all $k \in \mathbb{N}$. We can easily show that $\{y_k\}$ is a Cauchy sequence in $Fix(\mathcal{F})$ and so it converges to a point $p$. Since $F(\mathcal{F})$ is closed, therefore $p \in Fix(\mathcal{F})$ and $\{x_{n_k}\}$ converges strongly to $p$. Since $\lim_{n\to\infty} d(x_n, p)$ exists, we have that $x_n \to p$. The proof is completed. □

## 4. Example

Now we give the example of $\mathcal{F} : \mathcal{K} \to \mathcal{K}$ be a $C$-$\alpha$ nonexpansive mapping with $\alpha \in [0, 1)$ which is not generalized $\alpha-$nonexpansive mappings.

**Example 4.1.** *Let $\mathcal{K} = [0, 4] \subset \mathbb{R}$ endowed with usual norm in $\mathbb{R}$. Define a mapping $\mathcal{F} : \mathcal{K} \to \mathcal{K}$ by*

$$\mathcal{F}x = \begin{cases} \frac{x}{5}, x \neq 4 \\ \frac{16}{5}, x = 4 \end{cases}$$

*To verify that for $\alpha = \frac{3}{4}$, $\mathcal{F}$ is a C-$\frac{3}{4}$ nonexpansive mapping, we consider the following cases:*
***Case I:****If $x, y \neq 4$, then*

$$\begin{aligned}
& \alpha\left|\mathcal{F}x - y\right|^2 + \alpha\left|\mathcal{F}y - x\right|^2 + (1-2\alpha)\left|x-y\right|^2 \\
= & \frac{3}{4}\left|\mathcal{F}x - y\right|^2 + \frac{3}{4}\left|\mathcal{F}y - x\right|^2 - \frac{1}{2}\left|x-y\right|^2 \\
= & \frac{3}{4}(\frac{1}{5}x - y)^2 + \frac{3}{4}(\frac{1}{5}y - x)^2 - \frac{1}{2}(x-y)^2 \\
= & \frac{3}{4}(\frac{1}{25}x^2 - \frac{2}{5}xy + y^2) + \frac{3}{4}(\frac{1}{25}y^2 - \frac{2}{5}xy + x^2) - \frac{1}{2}x^2 + xy - \frac{1}{2}y^2 \\
= & \frac{3}{100}x^2 - \frac{6}{20}xy + \frac{3}{4}y^2 + \frac{3}{100}y^2 - \frac{6}{20}xy + \frac{3}{4}x^2 - \frac{1}{2}x^2 + xy - \frac{1}{2}y^2 \\
= & (\frac{1}{5}x - \frac{1}{5}y)^2 + \frac{6}{25}x^2 + \frac{6}{25}y^2 + \frac{12}{25}xy \geq \left|\frac{1}{5}x - \frac{1}{5}y\right|^2 = \left|\mathcal{F}x - \mathcal{F}y\right|^2.
\end{aligned}$$

*Since for $x, y \in [0, 4)$, $\frac{6}{25}x^2 + \frac{6}{25}y^2 + \frac{12}{25}xy \geq 0$, then $\mathcal{F}$ is a C-$\frac{3}{4}$ nonexpansive mapping.*
***Case II:****If $x = 4, y \neq 4$, then*

$$\begin{aligned}
& \alpha\left|\mathcal{F}x - y\right|^2 + \alpha\left|\mathcal{F}y - x\right|^2 + (1-2\alpha)\left|x-y\right|^2 \\
= & \frac{3}{4}\left|\mathcal{F}x - y\right|^2 + \frac{3}{4}\left|\mathcal{F}y - x\right|^2 - \frac{1}{2}\left|x-y\right|^2 \\
= & \frac{3}{4}(\frac{16}{5} - y)^2 + \frac{3}{4}(\frac{1}{5}y - 4)^2 - \frac{1}{2}(4-y)^2 \\
= & \frac{3}{4}(\frac{256}{25} - \frac{32}{5}y + y^2) + \frac{3}{4}(\frac{1}{25}y^2 - \frac{8}{5}y + 16) - \frac{16}{2} + 8y - \frac{1}{2}y^2 \\
= & (\frac{16}{5} - \frac{1}{5}y)^2 + \frac{6}{25}y^2 + \frac{82}{25}y + \frac{36}{25} \geq \left|\frac{16}{5} - \frac{1}{5}y\right|^2 = \left|\mathcal{F}x - \mathcal{F}y\right|^2.
\end{aligned}$$

*Since for* $y \in [0,4)$, $\frac{6}{25}y^2 + \frac{82}{25}y + \frac{36}{25} \geq 0$, *then* $\mathcal{F}$ *is a C-*$\frac{3}{4}$ *nonexpansive mapping. Contrarily at* $x = 4, y = 3$*; we get*

$$\frac{1}{2}|x - \mathcal{F}x| = \frac{1}{2}\left|4 - \frac{16}{5}\right| = \frac{2}{5} \leq 1 = |x - y|.$$

*Then, we have*

$$\begin{aligned} & \alpha|\mathcal{F}x - y| + \alpha|\mathcal{F}y - x| + (1-2\alpha)|x-y| \\ = & \alpha\left|\frac{16}{5} - 3\right| + \alpha\left|\frac{3}{4} - 4\right| + (1-2\alpha)|4-3| \\ = & \frac{1}{5}\alpha + \frac{17}{5}\alpha + 1 - 2\alpha = 1 + \frac{8}{5}\alpha \\ < & \left|\frac{16}{5} - \frac{3}{5}\right| = \frac{13}{5} = 1 + \frac{8}{5} = |\mathcal{F}x - \mathcal{F}y|. \end{aligned}$$

*Hence* $\mathcal{F}$ *is not a generalized* $\alpha-$*nonexpansive mapping for* $\alpha = \frac{3}{4}$.

We now compare convergence behavior of SP*-iteration process with other iteration processes using Example 4.1. From Figure 1, we see that the SP*-iteration process converges faster than SP-iteration and Picard Normal S-iteration processes. Let $\{a_n\} = \{b_n\} = \{c_n\} = 0.8$ and initial point be $x_1 = 4$. The fixed point of the mapping defined in Example 4.1 is $0$. These can be seen in Figure 1.

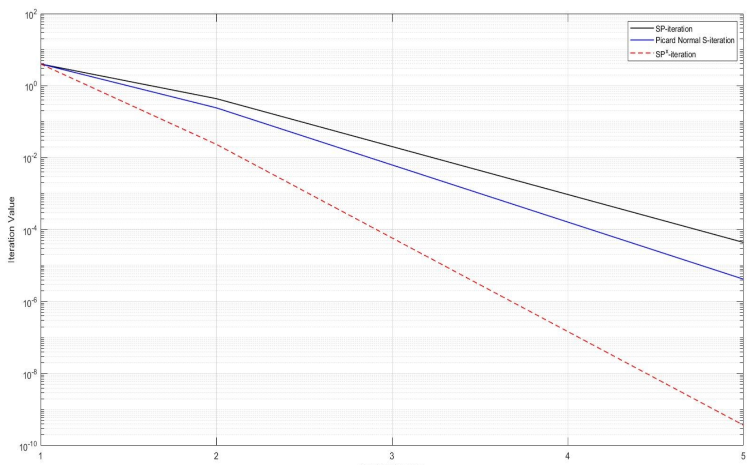


FIGURE 1. Convergence of SP-iteration, Picard Normal S-iteration and SP*-iteration processes to the fixed point 0 of the mapping defined in Example 4.1.

## 5. DATA DEPENDENCE OF SP*-ITERATION PROCESS

In 2021, Temir and Korkut [31] introduced the iterative process generated by (1.6) (SP*-iteration process) and they established that the rate of convergence of the SP*-iteration process is faster than the SP-iteration process and the Picard Normal S-iteration process. Also Temir [29] proved the stability of the SP*-iteration process in $CAT(0)$ spaces. In this paper, we also prove data depence result of the SP*-iterative process. First, we give the following strong convergence theorem.

**Theorem 5.8.** [29] *Let $\mathcal{K}$ be a nonempty closed convex subset of a complete $CAT(0)$ space $\mathcal{X}$ , $\mathcal{F}:\mathcal{X}\to\mathcal{X}$ be a contraction mapping with $Fix(\mathcal{F})\neq\emptyset$. Let $\{x_n\}$ be a sequence generated by* (1.6) *with real sequences $\{c_n\},\{bn\}$ and $\{a_n\}\in[0,1]$ with $\sum_{n=1}^{\infty}a_n=\infty$. Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to an unique fixed point of $\mathcal{F}$.*

In what follows, we shall make use of the following well-known lemma.

**Lemma 5.11.** *(*[28]*) Let $\{\kappa_n\}$ be a nonnegative sequence for which one assumes that there exists an $n_0\in\mathbb{N}$ such that, for all $n\geq n_0$,*

$$\kappa_{n+1}\leq(1-\vartheta_n)\kappa_n+\vartheta_n\sigma_n$$

*is satisfied, where $\vartheta_n\in(0,1)$ for all $n\in\mathbb{N}$, $\sum_{n=0}^{\infty}\vartheta_n=\infty$ and $\sigma_n\geq 0$, $\forall n\in\mathbb{N}$. Then the following holds: $0\leq\limsup_{n\to\infty}\kappa_n\leq\limsup_{n\to\infty}\sigma_n$.*

**Definition 5.5.** *(*[2]*) Let $\mathcal{F},\widetilde{\mathcal{F}}:\mathcal{X}\to\mathcal{X}$ be two operators. We say that $\widetilde{\mathcal{F}}$ is an approximate operator for $\mathcal{F}$ if, for all $x\in X$ and for a fixed $\epsilon>0$, we have $d(\mathcal{F}x,\widetilde{\mathcal{F}}x)\leq\epsilon$.*

By using this definition, we now prove the data dependence result for the iteration process defined by (1.6).

**Theorem 5.9.** *Suppose $\mathcal{K},\mathcal{X},\mathcal{F}$ are as in Theorem 5.8. Consider $\widetilde{\mathcal{F}}$ being an approximate operator for the contraction mapping $\mathcal{F}$ with possible numerical error $\epsilon>0$. Further, $\{x_n\}$ is the sequence* (1.6)*, for $\mathcal{F}$, and define approximate sequence $\omega_n$ for $\widetilde{\mathcal{F}}$ as follows:*

$$\left\{\begin{array}{l}\tau_n=\widetilde{\mathcal{F}}((1-c_n)\omega_n\oplus c_n\widetilde{\mathcal{F}}\omega_n),\\ \upsilon_n=\widetilde{\mathcal{F}}((1-b_n)\tau_n\oplus b_n\widetilde{\mathcal{F}}\tau_n),\\ \omega_{n+1}=\widetilde{\mathcal{F}}((1-a_n)\upsilon_n\oplus a_n\widetilde{\mathcal{F}}\upsilon_n)\forall n\in\mathbb{N},\end{array}\right. \tag{5.10}$$

*where $\mathcal{K}$ is a nonempty closed convex subset of a $CAT(0)$ space, $x_1\in\mathcal{K}$, $\{a_n\}$, $\{b_n\}$ and $\{c_n\}\in[0,1]$ satisfying the condition; $\frac{1}{2}\leq a_n,\forall n\in\mathbb{N}$.*

*If $\mathcal{F}p=p$ and $\widetilde{\mathcal{F}}\widetilde{p}=\widetilde{p}$ such that $\lim_{n\to\infty}\omega_n=\widetilde{p}$ then we have $d(p,\widetilde{p})\leq\frac{11\epsilon}{1-\theta}$.*

*Proof.* Let us consider (1.6) and (5.10), we have

$$\begin{aligned}
d(z_n,\tau_n) &\leq d(\mathcal{F}((1-c_n)x_n\oplus c_n\mathcal{F}x_n),\widetilde{\mathcal{F}}((1-c_n)\omega_n\oplus c_n\widetilde{\mathcal{F}}\omega_n))\\
&\leq d(\mathcal{F}((1-c_n)x_n\oplus c_n\mathcal{F}x_n),\mathcal{F}((1-c_n)\omega_n\oplus c_n\widetilde{\mathcal{F}}\omega_n))\\
&\quad + d(\mathcal{F}((1-c_n)\omega_n\oplus c_n\widetilde{\mathcal{F}}\omega_n),\widetilde{\mathcal{F}}((1-c_n)\omega_n\oplus c_n\widetilde{\mathcal{F}}\omega_n))\\
&\leq \theta[d((1-c_n)x_n\oplus c_n\mathcal{F}x_n,(1-c_n)\omega_n\oplus c_n\widetilde{\mathcal{F}}\omega_n)]+\epsilon\\
&\leq \theta[(1-c_n)d(x_n,\omega_n)+c_nd(\mathcal{F}x_n,\widetilde{\mathcal{F}}\omega_n)]+\epsilon\\
&\leq \theta[(1-c_n)d(x_n,\omega_n)+c_nd(\mathcal{F}x_n,\mathcal{F}\omega_n)+d(\mathcal{F}\omega_n,\widetilde{\mathcal{F}}\omega_n)]+\epsilon\\
&\leq \theta[(1-c_n)d(x_n,\omega_n)+\theta c_nd(x_n,\omega_n)+c_n\epsilon]+\epsilon\\
&\leq \theta[(1-c_n(1-\theta))d(x_n,\omega_n)+c_n\epsilon]+\epsilon. \qquad (5.11)
\end{aligned}$$

By using (1.6) and (5.11), we have

$$
\begin{aligned}
d(y_n, \upsilon_n) &\leq d(\mathcal{F}((1-b_n)z_n \oplus b_n\mathcal{F}z_n), \widetilde{\mathcal{F}}((1-b_n)\tau_n \oplus c_n\widetilde{\mathcal{F}}\tau_n)) \\
&\leq d(\mathcal{F}((1-b_n)z_n \oplus b_n\mathcal{F}z_n), \mathcal{F}((1-b_n)\tau_n \oplus b_n\widetilde{\mathcal{F}}\tau_n)) \\
&+ d(\mathcal{F}((1-b_n)\tau_n \oplus b_n\widetilde{\mathcal{F}}\tau_n), \widetilde{\mathcal{F}}((1-b_n)\tau_n \oplus b_n\widetilde{\mathcal{F}}\tau_n)) \\
&\leq \theta[d((1-b_n)z_n \oplus b_n\mathcal{F}z_n, (1-b_n)\tau_n \oplus b_n\widetilde{\mathcal{F}}\tau_n)] + \epsilon \\
&\leq \theta[(1-b_n)d(z_n, \tau_n) + b_n d(\mathcal{F}z_n, \widetilde{\mathcal{F}}\tau_n)] + \epsilon \\
&\leq \theta[(1-b_n)d(z_n, \tau_n) + b_n d(\mathcal{F}z_n, \mathcal{F}\tau_n) + d(\mathcal{F}\tau_n, \widetilde{\mathcal{F}}\tau_n)] + \epsilon \\
&\leq \theta[(1-b_n)d(z_n, \tau_n) + \theta b_n d(z_n, \tau_n) + \epsilon] + \epsilon \\
&\leq \theta[(1-b_n(1-\theta))d(z_n, \tau_n) + b_n\epsilon] + \epsilon \\
&\leq \theta^2[(1-b_n(1-\theta))(1-c_n(1-\theta))d(x_n, \omega_n) + c_n\epsilon(1-b_n(1-\theta))] \\
&+ \theta b_n\epsilon + \theta\epsilon + \epsilon \\
&\leq (1-b_n(1-\theta))(1-c_n(1-\theta))d(x_n, \omega_n) + c_n\epsilon(1-b_n(1-\theta)) \qquad (5.12) \\
&+ \theta b_n\epsilon + \theta\epsilon + \epsilon.
\end{aligned}
$$

By using (1.6) and (5.12), we have

$$
\begin{aligned}
d(x_{n+1}, \omega_{n+1}) &\leq d(\mathcal{F}((1-a_n)y_n \oplus a_n\mathcal{F}\upsilon_n), \widetilde{\mathcal{F}}((1-a_n)\upsilon_n \oplus a_n\widetilde{\mathcal{F}}\upsilon_n)) \\
&\leq d(\mathcal{F}((1-a_n)y_n \oplus a_n\mathcal{F}y_n), \mathcal{F}((1-a_n)\upsilon_n \oplus a_n\widetilde{\mathcal{F}}\upsilon_n)) \\
&+ d(\mathcal{F}((1-a_n)\upsilon_n \oplus a_n\widetilde{\mathcal{F}}\upsilon_n), \widetilde{\mathcal{F}}((1-a_n)\upsilon_n \oplus a_n\widetilde{\mathcal{F}}\upsilon_n)) \\
&\leq \theta[d((1-a_n)\upsilon_n \oplus a_n\mathcal{F}y_n, (1-a_n)\upsilon_n \oplus a_n\widetilde{\mathcal{F}}\upsilon_n)] + \epsilon \\
&\leq \theta[(1-a_n)d(y_n, \upsilon_n) + a_n d(\mathcal{F}y_n, \widetilde{\mathcal{F}}\upsilon_n)] + \epsilon \\
&\leq \theta[(1-a_n)d(y_n, \upsilon_n) + a_n d(\mathcal{F}y_n, \mathcal{F}\upsilon_n) + d(\mathcal{F}\upsilon_n, \widetilde{\mathcal{F}}\upsilon_n)] + \epsilon \\
&\leq \theta[(1-a_n)d(y_n, \upsilon_n) + \theta b_n d(y_n, \upsilon_n) + \epsilon] + \epsilon \\
&\leq \theta[(1-a_n(1-\theta))d(y_n, \upsilon_n) + a_n\epsilon] + \epsilon \\
&\leq \theta(1-a_n(1-\theta))[(1-b_n(1-\theta))(1-c_n(1-\theta))d(x_n, \omega_n) \qquad (5.13) \\
&+ c_n\epsilon\theta(1-b_n(1-\theta)) + \theta b_n\epsilon + \theta\epsilon + a_n\epsilon + \epsilon] + \epsilon.
\end{aligned}
$$

Since $\{b_n\}, \{c_n\} \in [0,1]$ and $\theta \in [0,1)$, it implies that $(1-b_n(1-\theta)) < 1$ and $(1-c_n(1-\theta)) < 1$ and rearranging (5.13), we get

$$
\begin{aligned}
d(x_{n+1}, \omega_{n+1}) &\leq (1-a_n(1-\theta))d(x_n, \omega_n) \qquad (5.14) \\
&+ (1-a_n(1-\theta)) + a_n\epsilon + 5\epsilon.
\end{aligned}
$$

We note that $1 - a_n \leq a_n$, so we obtain

$$
\begin{aligned}
d(x_{n+1}, \omega_{n+1}) &\leq (1-a_n(1-\theta))d(x_n, \omega_n) + a_n\epsilon + 5\epsilon(1 - a_n + a_n) \qquad (5.15) \\
&\leq (1-a_n(1-\theta))d(x_n, \omega_n) + a_n(1-\theta)\frac{11\epsilon}{(1-\theta)}.
\end{aligned}
$$

Define

$$
\begin{aligned}
d(x_n, \omega_n) &= \kappa_n \\
a_n(1-\theta) &= \vartheta_n \\
\frac{11\epsilon}{(1-\theta)} &= \sigma_n
\end{aligned}
$$

Then from Lemma 5.11 and (5.15), we obtain

$$0 \leq \limsup_{n\to\infty} d(x_n, \omega_n) \leq \limsup_{n\to\infty} \frac{11\epsilon}{(1-\theta)}. \tag{5.16}$$

By Theorem 5.8, we have $\lim_{n\to\infty} x_n = p$, and by assumption, we have $\lim_{n\to\infty} \omega_n = \widetilde{p}$. By using these facts together with (5.16), we obtain

$$d(p, \widetilde{p}) \leq \frac{11\epsilon}{1-\theta}.$$

□

## 6. Application to nonlinear integral equation

In this article, our interest is to approximate the solution of the following integral equation (6.19) via of SP*-iteration process. Let $C([a,b])$ denote the space of all continuous functions on the interval $[a,b]$, endowed with the metric, $d_\infty(x,y) = max_{s\in[a,b]}|x(s) - y(s)|; \forall x, y \in C([a,b])$. In this section we will be interested in the following delay differential equation:

$$\acute{x}(t) = \phi(t, x(t), x(t-\mu)), t \in [t_0, b], \tag{6.17}$$

with initial condition

$$x(t) = \psi(t), t \in [t_0 - \mu, t_0]. \tag{6.18}$$

To achieve our aim, the following axioms are considered:

(i) $t_0, b \in \mathbb{R}, \mu > 0$;
(ii) $\phi \in C([t_0, b] \times \mathbb{R}^2, \mathbb{R})$;
(iii) $\psi \in C([t_0 - \mu, b], \mathbb{R})$;
(iv) There exists $L_\phi > 0$, such that;

$$|\phi(t, x_1, x_2) - \phi(t, y_1, y_2)| \leq L_\phi(|x_1 - y_1| + |x_2 - y_2|, \forall x_i, y_i \in \mathbb{R}, i = 1, 2, t \in [t_0, b];$$

(v) $2L_\phi(b - t_0) < 1$.

Next, the (6.17) and (6.18) are reformulated as the following integral equation:

$$x(t) = \begin{cases} \psi(t), & t \in [t_0 - \mu, t_0] \\ \psi(t_0) + \int_{t_0}^{t} \phi(s, x(s), x(s-\mu))ds, & t \in [t_0, b]. \end{cases} \tag{6.19}$$

The following result can be found in [7].

**Theorem 6.10.** *Assume that conditions (i)-(v) are hold. Then the problem* (6.17)-(6.18) *has a unique solution, say* $p \in C([t_0 - \mu, b], \mathbb{R}) \cap C^1([t_0, b], \mathbb{R})$ *and* $p = \lim_{n\to\infty} T^n(x)$ *for any* $x \in C([t_0 - \mu, b], \mathbb{R})$.

Next, we will prove that SP*-iteration process converges strongly to the unique solution of integral equation (6.19). For this, we give our main result in this section as follows:

**Theorem 6.11.** *Assume that conditions (i)-(v) are hold. Then the problem* (6.17)-(6.18) *has a unique solution, say* $p \in C([t_0 - \mu, b], \mathbb{R}) \cap C^1([t_0, b], \mathbb{R})$. *Let* $\{x_n\}$ *be a sequence generated by the algorithm* (1.6) *with real sequences* $\{c_n\}, \{b_n\}$ *and* $\{a_n\} \in [0,1]$ *with* $\sum_{n=1}^{\infty} a_n = \infty$. *Then* $\{x_n\}$ *converges to* $p$.

*Proof.* Let

$$\mathcal{F}x(t) = \begin{cases} \psi(t), & t \in [t_0 - \mu, t_0] \\ \psi(t_0) + \int_{t_0}^{t} \phi(s, x(s), x(s-\mu))ds, & t \in [t_0, b]. \end{cases} \tag{6.20}$$

be an integral operator with respect to (6.19). Let $\{x_n\}$ be a sequence defined by the SP*-iteration process for the operator (6.20). Denote by $p$ be the fixed point of $\mathcal{F}$. We want to show $x_n \to p$ as $n \to \infty$. For $t \in [t_0-\mu, t_0]$, it is easy to see that $x_n \to p$ as $n \to \infty$. Assume $t \in [t_0, b]$. We define $\eta_n = (1-c_n)x_n \oplus c_n \mathcal{F}x_n;\ z_n = \mathcal{F}\eta_n;\ \tau_n = (1-b_n)z_n \oplus b_n\mathcal{F}z_n;\ y_n = \mathcal{F}\tau_n;\ \xi_n = (1-a_n)y_n \oplus a_n\mathcal{F}y_n;\ x_{n+1} = \mathcal{F}\xi_n, n \in \mathbb{N}$. Using (1.6) and $(iv)$, we obtain

$$\begin{aligned}
d_\infty(\eta_n, p) &= d_\infty(((1-c_n)x_n \oplus c_n\mathcal{F}x_n), p) \\
&\leq (1-c_n)d_\infty(x_n,p) + c_n d_\infty(\mathcal{F}x_n, p) \\
&\leq (1-c_n)d_\infty(x_n,p) \\
&+ max_{t\in[t_0-\mu,b]}\Big|\psi(t_0) + c_n\int_{t_0}^{t}\phi(s, x_n(s), x_n(s-\mu))ds \\
&- \psi(t_0) - \int_{t_0}^{t}\phi(s,p(s),p(s-\mu))ds\Big| \\
&\leq (1-c_n)d_\infty(x_n,p) \\
&+ c_n max_{t\in[t_0-\mu,b]}\int_{t_0}^{t}|\phi(s,x_n(s),x_n(s-\mu))ds \\
&- \phi(s,p(s),p(s-\mu))ds| \\
&\leq (1-c_n)d_\infty(x_n,p) \\
&+ c_n max_{t\in[t_0-\mu,b]}\int_{t_0}^{t}L_\phi(|x_n(s)-p(s)| \\
&+ |x_n(s-\mu)-p(s-\mu)|)ds \\
&\leq (1-c_n)d_\infty(x_n,p) \\
&+ c_n\int_{t_0}^{t}L_\phi max_{t\in[t_0-\mu,b]}(|x_n(s)-p(s)| \\
&+ max_{t\in[t_0-\mu,b]}|x_n(s-\mu)-p(s-\mu)|)ds \\
&\leq (1-c_n)d_\infty(x_n,p) + c_n\int_{t_0}^{t}L_\phi(d_\infty(x_n,p) \\
&+ d_\infty(x_n,p))ds \\
&\leq (1-c_n)d_\infty(x_n,p) + 2c_nL_\phi(b-t_0)d_\infty(x_n,p) \\
&\leq (1-c_n(1-2L_\phi(b-t_0)))d_\infty(x_n,p).
\end{aligned}$$

Thus,

$$d_\infty(\eta_n,p) \leq (1-c_n(1-2L_\phi(b-t_0)))d_\infty(x_n,p). \tag{6.21}$$

Using (1.6), $(iv)$ and (6.21), we obtain

$$\begin{aligned}
d_\infty(z_n,p) &= d_\infty(\mathcal{F}\eta_n,p) = d_\infty(\mathcal{F}\eta_n,\mathcal{F}p) \\
&= max_{t\in[t_0-\mu,b]}\big|\mathcal{F}\eta_n - \mathcal{F}p\big| \\
&= max_{t\in[t_0-\mu,b]}\Big|\psi(t_0) + \int_{t_0}^{t}\phi(s,\eta_n(s),\eta_n(s-\mu))ds \\
&- \psi(t_0) - \int_{t_0}^{t}\phi(s,p(s),p(s-\mu))ds\Big|
\end{aligned}$$

$$
\begin{aligned}
&= max_{t\in[t_0-\mu,b]}\Big|\int_{t_0}^{t}\phi(s,\eta_n(s),\eta_n(s-\mu))ds\\
&- \phi(s,p(s),p(s-\mu))ds\Big|\\
&\leq max_{t\in[t_0-\mu,b]}\int_{t_0}^{t}L_\phi\big(|\eta_n(s)-p(s)|\\
&+ |\eta_n(s-\mu)-p(s-\mu)|\big)ds\\
&\leq \int_{t_0}^{t}L_\phi\big(|max_{t\in[t_0-\mu,b]}|\eta_n(s)-p(s)|\\
&+ max_{t\in[t_0-\mu,b]}|\eta_n(s-\mu)-p(s-\mu)|\big)ds\\
&= \int_{t_0}^{t}L_\phi\big(d_\infty(\eta_n(s),p(s))+d_\infty(\eta_n(s-\mu),p(s-\mu))\big)ds\\
&= \int_{t_0}^{t}L_\phi\big(d_\infty(\eta_n(s),p(s))+d_\infty(\eta_n(s),p(s))\big)ds\\
&\leq 2L_\phi(b-t_0)d_\infty(\eta_n(s),p(s))\\
&\leq 2L_\phi(b-t_0)(1-c_n(1-2L_\phi(b-t_0)))d_\infty(x_n,p).
\end{aligned}
$$

Thus,

$$
d_\infty(z_n,p)\leq 2L_\phi(b-t_0)(1-c_n(1-2L_\phi(b-t_0)))d_\infty(x_n,p). \tag{6.22}
$$

Using (1.6) and $(iv)$, we obtain

$$
\begin{aligned}
d_\infty(\tau_n,p) &= d_\infty(((1-b_n)z_n\oplus b_n\mathcal{F}z_n),p)\\
&\leq (1-b_n)d_\infty(z_n,p)+b_nd_\infty(\mathcal{F}z_n,p)\\
&\leq (1-b_n)d_\infty(z_n,p)\\
&+ max_{t\in[t_0-\mu,b]}\Big|\psi(t_0)+b_n\int_{t_0}^{t}\phi(s,z_n(s),z_n(s-\mu))ds\\
&- \psi(t_0)-\int_{t_0}^{t}\phi(s,p(s),p(s-\mu))ds\Big|\\
&\leq (1-b_n)d_\infty(z_n,p)\\
&+ b_nmax_{t\in[t_0-\mu,b]}\int_{t_0}^{t}|\phi(s,z_n(s),z_n(s-\mu))ds\\
&- \phi(s,p(s),p(s-\mu))ds|\\
&\leq (1-b_n)d_\infty(z_n,p)\\
&+ b_nmax_{t\in[t_0-\mu,b]}\int_{t_0}^{t}L_\phi(|z_n(s)-p(s)|\\
&+ |z_n(s-\mu)-p(s-\mu)|)ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq (1-b_n)d_\infty(z_n,p)\\
&+ b_n\int_{t_0}^{t} L_\phi max_{t\in[t_0-\mu,b]}(|z_n(s)-p(s)|\\
&+ max_{t\in[t_0-\mu,b]}|z_n(s-\mu)-p(s-\mu)|)ds\\
&\leq (1-b_n)d_\infty(z_n,p)+b_n\int_{t_0}^{t} L_\phi(d_\infty(z_n,p)\\
&+ d_\infty(z_n,p))ds\\
&\leq (1-b_n)d_\infty(z_n,p)+2b_nL_\phi(b-t_0)d_\infty(z_n,p)\\
&\leq (1-b_n(1-2L_\phi(b-t_0)))d_\infty(z_n,p).
\end{aligned}
$$

Thus,

$$
d_\infty(\tau_n,p)\leq (1-b_n(1-2L_\phi(b-t_0)))d_\infty(z_n,p). \tag{6.23}
$$

Using (1.6), $(iv)$ and (6.23), we obtain

$$
\begin{aligned}
d_\infty(y_n,p) &= d_\infty(\mathcal{F}\tau_n,p)=d_\infty(\mathcal{F}\tau_n,\mathcal{F}p)\\
&= max_{t\in[t_0-\mu,b]}\big|\mathcal{F}\tau_n-\mathcal{F}p\big|\\
&= max_{t\in[t_0-\mu,b]}\big|\psi(t_0)+\int_{t_0}^{t}\phi(s,\tau_n(s),\tau_n(s-\mu))ds\\
&- \psi(t_0)-\int_{t_0}^{t}\phi(s,p(s),p(s-\mu))ds\big|\\
&= max_{t\in[t_0-\mu,b]}\big|\int_{t_0}^{t}\phi(s,\tau_n(s),\tau_n(s-\mu))ds\\
&- \phi(s,p(s),p(s-\mu))ds\big|\\
&\leq max_{t\in[t_0-\mu,b]}\int_{t_0}^{t}L_\phi\big(|\tau_n(s)-p(s)|\\
&+ |\tau_n(s-\mu)-p(s-\mu)|\big)ds\\
&\leq \int_{t_0}^{t}L_\phi\big(|max_{t\in[t_0-\mu,b]}|\tau_n(s)-p(s)|\\
&+ max_{t\in[t_0-\mu,b]}|\tau_n(s-\mu)-p(s-\mu)|\big)ds.
\end{aligned}
$$

Then we get

$$
\begin{aligned}
d_\infty(y_n,p) &\leq \int_{t_0}^{t}L_\phi\big(d_\infty(\tau_n(s),p(s))+d_\infty(\tau_n(s-\mu),p(s-\mu))\big)ds\\
&= \int_{t_0}^{t}L_\phi\big(d_\infty(\tau_n(s),p(s))+d_\infty(\tau_n(s),p(s))\big)ds\\
&\leq 2L_\phi(b-t_0)d_\infty(\tau_n(s),p(s))\\
&\leq 2L_\phi(b-t_0)(1-b_n(1-2L_\phi(b-t_0)))d_\infty(z_n,p).
\end{aligned}
$$

Thus,

$$
d_\infty(y_n,p)\leq 2L_\phi(b-t_0)(1-b_n(1-2L_\phi(b-t_0)))d_\infty(z_n,p). \tag{6.24}
$$

Using (1.6) and $(iv)$, we obtain

$$\begin{aligned}
d_\infty(\xi_n,p) &= d_\infty(((1-a_n)y_n \oplus b_n\mathcal{F}y_n),p)\\
&\leq (1-a_n)d_\infty(y_n,p)+a_nd_\infty(\mathcal{F}y_n,p)\\
&\leq (1-a_n)d_\infty(y_n,p)\\
&+ max_{t\in[t_0-\mu,b]}\big|\psi(t_0)+a_n\int_{t_0}^{t}\phi(s,y_n(s),y_n(s-\mu))ds\\
&- \psi(t_0)-\int_{t_0}^{t}\phi(s,p(s),p(s-\mu))ds\big|\\
&\leq (1-a_n)d_\infty(y_n,p)\\
&+ a_nmax_{t\in[t_0-\mu,b]}\int_{t_0}^{t}|\phi(s,y_n(s),y_n(s-\mu))ds\\
&- \phi(s,p(s),p(s-\mu))ds|\\
&\leq (1-a_n)d_\infty(y_n,p)\\
&+ a_nmax_{t\in[t_0-\mu,b]}\int_{t_0}^{t}L_\phi(|y_n(s)-p(s)|\\
&+ |y_n(s-\mu)-p(s-\mu)|)ds\\
&\leq (1-a_n)d_\infty(y_n,p)\\
&+ a_n\int_{t_0}^{t}L_\phi max_{t\in[t_0-\mu,b]}(|y_n(s)-p(s)|\\
&+ max_{t\in[t_0-\mu,b]}|y_n(s-\mu)-p(s-\mu)|)ds\\
&\leq (1-a_n)d_\infty(y_n,p)+a_n\int_{t_0}^{t}L_\phi(d_\infty(y_n,p)\\
&+ d_\infty(y_n,p))ds\\
&\leq (1-a_n)d_\infty(y_n,p)+2b_nL_\phi(b-t_0)d_\infty(y_n,p)\\
&\leq (1-a_n(1-2L_\phi(b-t_0)))d_\infty(y_n,p).
\end{aligned}$$

Thus,

$$d_\infty(\xi_n,p)\leq(1-a_n(1-2L_\phi(b-t_0)))d_\infty(y_n,p). \tag{6.25}$$

Using (1.6), $(iv)$ and (6.25), we obtain

$$\begin{aligned}
d_\infty(x_{n+1},p) &= d_\infty(\mathcal{F}\xi_n,p)=d_\infty(\mathcal{F}\xi_n,\mathcal{F}p)\\
&= max_{t\in[t_0-\mu,b]}\big|\mathcal{F}\xi_n-\mathcal{F}p\big|\\
&= max_{t\in[t_0-\mu,b]}\big|\psi(t_0)+\int_{t_0}^{t}\phi(s,\xi_n(s),\xi_n(s-\mu))ds\\
&- \psi(t_0)-\int_{t_0}^{t}\phi(s,p(s),p(s-\mu))ds\big|\\
&= max_{t\in[t_0-\mu,b]}\big|\int_{t_0}^{t}\phi(s,\xi_n(s),\xi_n(s-\mu))ds\\
&- \phi(s,p(s),p(s-\mu))ds\big|\\
&\leq max_{t\in[t_0-\mu,b]}\int_{t_0}^{t}L_\phi\big(|\xi_n(s)-p(s)|\\
&+ |\xi_n(s-\mu)-p(s-\mu)|\big)ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_{t_0}^{t} L_\phi\big(|max_{t\in[t_0-\mu,b]}|\xi_n(s)-p(s)| \\
&+ max_{t\in[t_0-\mu,b]}|\xi_n(s-\mu)-p(s-\mu)|\big)ds \\
&= \int_{t_0}^{t} L_\phi\big(d_\infty(\xi_n(s),p(s))+d_\infty(\xi_n(s-\mu),p(s-\mu))\big)ds \\
&= \int_{t_0}^{t} L_\phi\big(d_\infty(\xi_n(s),p(s))+d_\infty(\xi_n(s),p(s))\big)ds \\
&\leq 2L_\phi(b-t_0)d_\infty(\xi_n(s),p(s)) \leq 2L_\phi(b-t_0)(1-a_n(1-2L_\phi(b-t_0)))d_\infty(y_n,p).
\end{aligned}
$$

Thus, from (6.22), (6.24) we obtain

$$
\begin{aligned}
d_\infty(x_{n+1},p) \quad &\leq \quad (2L_\phi(b-t_0))^3(1-a_n(1-2L_\phi(b-t_0))) \\
&\quad (1-b_n(1-2L_\phi(b-t_0))) \\
&\quad (1-c_n(1-2L_\phi(b-t_0)))d_\infty(x_n,p). \qquad (6.26)
\end{aligned}
$$

From assumption $(v)$ and the fact that $(1-c_n(1-2L_\phi(b-t_0)))<1$ and $(1-b_n(1-2L_\phi(b-t_0)))<1$, (6.26) reduces

$$d_\infty(x_{n+1},p) \leq (1-a_n(1-2L_\phi(b-t_0)))d_\infty(x_n,p).. \qquad (6.27)$$

Via induction, we obtain

$$d_\infty(x_{n+1},p) \quad \leq \quad \prod_{k=1}^{n}[1-a_k((1-2L_\phi(b-t_0)))]d_\infty(x_1,p).$$

So, we know that $1-x \leq e^{-x}$ for all $x\in[0,1]$. Hence we have

$$d_\infty(x_{n+1},p) \leq e^{-(1-2L_\phi(b-t_0))\sum_{k=1}^{n}a_k}d_\infty(x_1,p).$$

Taking the limit of both sides of the above inequality , $x_n \to p$ as $n\to\infty$. Hence, the proof is complete. □

## 7. Conclusions

We obtain some results on the strong and $\Delta$-convergence of SP*-iteration process (1.6) for $C$-$\alpha$ nonexpansive mappings in nonlinear $CAT(0)$ spaces. The result herein complements the some results of [23, 30] from linear setting to $CAT(0)$ spaces. We also prove the data depence result of SP*-iteration process generated by (1.6) in this paper. In addition, we give an illustrative numerical example that satisfies $C$-$\alpha$ nonexpansive mapping. As seen in Example 4.1, the mapping is not a generalized $\alpha-$nonexpansive mapping. Further we apply to the approximate the solution of the integral equation. Lastly, in future studies, iteration process can be developed and iteration that converges faster than prominent iterations can be presented.

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## References

[1] Aoyama, K.; Kohsaka, F. Fixed point theorem for $\alpha$ -nonexpansive mappings in Banach spaces. *Nonlinear Analy.* **74** (2011), no. 13, 4378-4391.

[2] Berinde, V. *Iterative approximation of fixed points*. Second edition. Lecture Notes in Mathematics, 1912. Springer, Berlin, 2007.

[3] Basarır, M. ; Şahin, A. On the strong and $\Delta$-convergence of S-iteration process for generalized nonexpansive mappings on $CAT(0)$ space. *Thai J. Math.* **12** (2014), no. 3, 549-559.

[4] Bruhat, F. ; Tits, J. Groupes reductifs sur un corps local. *I. Donnees radicielles valuees Inst Hautes Etudes Sci Publ Math.* **41** (1972) 5-251. doi:10.1007/BF02715544.
[5] Bridson,M.; Haefliger, A. *Metric spaces of non-positive curvature*. Springer-Verlag, Berlin, Heidelberg, 1999.
[6] Burago, D.; Burago, Y.; Ivanov, S. *A course in metric geometry*. in:Graduate Studies in Math. **vol. 33**, Amer. Math. Soc., Providence, RI, 2001.
[7] Coman, G. H.; Pavel, G. Rus, I. and Rus, I. A. *Introduction in the theory of operatorial equation*. Ed. Dacia, Cluj-Napoca, 1976.
[8] Dhompongsa, S; Panyanak, B. On $\Delta$-convergence theorems in $CAT(0)$ spaces. *Comput. Math. Appl.* **56** (2008), 2572-2579.
[9] Dhompongsa, S; Kaewkhao, A.; Panyanak, B. Lim's theorems for multivalued mappings in $CAT(0)$ spaces. *J. Math. Anal. Appl.* **312** (2005), 478-487.
[10] Dhompongsa, S.; Kirk, W. A.; Panyanak, B. Nonexpansive set-valued mappings in metric and Banach spaces. *J. Nonlinear Convex Anal.* **8** (2007), 35-45.
[11] Dhompongsa, S.; Kirk, W. A.; Sims, B. Fixed points of uniformly lipschitzian mappings. *Nonlinear Anal. TMA* **65** (2006), 762-772.
[12] Fujiwara, K.; Nagano, K.; Shioya, T. Fixed point sets of parabolic isometries of $CAT(0)$spaces. *Comment. Math. Helv.* **81** (2006), 305-335.
[13] Gromov, M. *Metric structures for Riemannian and non-Riemannian spaces*. in: Progress in Mathematics, vol. 152. Birkhäuser, Boston, 1999.
[14] Kadıoglu N.; Yıldırım, I. Approximating fixed points of nonexpansive mappings by faster iteration process. arXiv preprint, (2014), arXiv:1402.6530.
[15] Kirk, W. A. *Geodesic geometry and fixed point theory*. in: Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003), in: Colecc. Abierta, vol. 64, pp. 195-225, Univ. Sevilla Secr. Publ., Seville, 2003.
[16] Kirk, W. A. *Geodesic geometry and fixed point theory II*. in: International Conference on Fixed Point Theory and Applications, Yokohama Publ., Yokohama, 2004, pp. 113-142.
[17] Kirk, W. A. Fixed point theorems in $CAT(0)$ spaces and R-trees. *Fixed Point Theory Appl.* (2004) 309-316.
[18] Kirk, W. A.; Panyanak, B. A concept of convergence in geodesic spaces. *Nonlinear Anal. TMA* **68** (2008), 3689-3696.
[19] T. C. Lim, Remarks on some fixed point theorems. *Proc. Amer. Math. Soc.* **60** (1976), 179-182.
[20] Lin, L. J.; Chuang C. S. ; Yu, Z.T. Fixed point theorems and $\Delta-$convergence theorems for generalized hybrid mappings on $CAT(0)$ spaces. *Fixed Point Theory and Applications* (2011), 2011:49.
[21] Nanjaras, B.; Panyanak, B. Demiclosed principle for asymptotically nonexpansive mappings in $CAT(0)$ spaces. *Fixed Point Theory and Applications* (2010), Article ID 268780, 14 pages doi:10.1155/2010/268780.
[22] Pant, R.; Shukla, R. Approximating fixed points of generalized $\alpha$-nonexpansive mappings in Banach spaces.*Numer. Funct. Anal. Optim.* **38** (2017), no. 2, 248-266.
[23] Pant, R.; Shukla, R. Fixed point theorems for a new class of nonexpansive mappings. *Appl. Gen. Topol.* **23** (2022), no. 2, 377-390.
[24] Phuengrattana, W.; Suantai, S. On the rate of convergence of Mann, Ishikawa, Noor and SP-iterations for continuous functions on an arbitrary interval. *J. Comput. Appl. Math.* **235** (2011), 3006-3014.
[25] Senter, H. F.; Dotson W. G. Jr. Approximating fixed points of nonexpansive mappings. *Proc. Amer. Math. Soc.* **44** (1974), 375-380.
[26] Sato, T. An alternative proof of Berg and Nikolaev's characterization of $CAT(0)$-spaces via quadrilateral inequality. *Arch. Math.* **93** (2009), 487-490.
[27] Suzuki, T. Fixed point theorems and convergence theorems for some generalized nonexpansive mappings, *Journal of Mathematical Analysis and Appl.* **340** (2008), no. 2, 1088-1095.
[28] Şoltuz, Ş. M. ; T. Grosan, T. Data dependence for Ishikawa iteration when dealing with contractive like operators. *Fixed Point Theory Appl.* (2008) Article 242916, 7 pages.
[29] Temir, S. Convergence theorems for operators with Property $(E)$ in $CAT(0)$ spaces. *Journal of Nonlinear Analysis and Optimization: Theory and Applications*, (Accepted).
[30] Temir, S.; Bayram, Z. Approximating common fixed point of three $C$-$\alpha$ nonexpansive mappings. *Konuralp Journal of Mathematics* **11** (2023), no. 2, 184-194.
[31] Temir, S.; Korkut, O. Approximating fixed point of the new SP*-iteration for generalized $\alpha$-nonexpansive mappings in $CAT(0)$ spaces. *Journal of Nonlinear Analysis and Optimization: Theory and Appl.* **12** (2021) Issue 2, pp.83-93.

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