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Initial Value Problem for Hybrid Improved Conformable Fractional Differential Equations with Retardation and Anticipation

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ABSTRACT. This manuscript is devoted to proving some results concerning the existence of solutions for a class of initial value problem for nonlinear implicit fractional Hybrid differential equations and improved conformable fractional derivative. The result is based on a fixed point theorem due to Dhage. Further, an example is provided for the justification of our main result.

1. INTRODUCTION

The fractional calculus has long been an attractive research topic in functional space theory due to its applicability in the modelling and scientific understanding of natural phenomena. Indeed, several applications in viscoelasticity and electrochemistry have been investigated. Non-integer derivatives of fractional order have been successfully used to generalize the fundamental laws of nature. For more details, we recommend [1–3, 5–10, 18, 20–25], and its references.

In [17], Khalil *et al.* provided a unique concept of fractional derivative, which is a natural extension of the traditional first derivative. The conformable fractional derivative is natural, and it contains most of the features of the classical integral derivative, such as product rule, quotient rule, linearity, chain rule, and power rule, and it is very useful for modelling different physical problems. Indeed, several publications have been produced since that time, and various equations have been solved using that notion [4, 11, 19].

F. Gao and C. Chi recently stated in [14] that there are still shortcomings for the conformable derivative and suggested an improved conformable fractional derivative to solve this issue. This derivative has a closer physical behaviour than the conformable fractional derivative of Riemann-Liouville and Caputo fractional derivatives. This improved conformable fractional derivative has a high potential for simulating different physical problems that normally use Riemann-Liouville or Caputo fractional derivatives.

In this paper, we study the existence of solutions for the initial value problem with nonlinear implicit hybrid fractional differential equation involving the improved Caputo-type conformable fractional derivative with retarded and advanced arguments:

$$ {}_{0}^{C}\tilde{\mathcal{T}}_{\vartheta}\left(\frac{y(t)}{\lambda(t,y(t))}\right)=\Psi\left(t,y^{t}(\cdot),{}_{0}^{C}\tilde{\mathcal{T}}_{\vartheta}\left(\frac{y(t)}{\lambda(t,y(t))}\right)\right),\ t\in(0,\mathcal{K}], \tag{1.1}$$

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$$y(t)=\xi(t),\ t\in[-\zeta,0],\ \zeta>0, \tag{1.2}$$

$$y(t)=\tilde{\xi}(t),\ t\in[\mathcal{K},\mathcal{K}+\delta],\ \delta>0, \tag{1.3}$$

where $0<\vartheta<1$, ${}_0^C\tilde{\mathcal{T}}_\vartheta$ is the improved Caputo-type conformable fractional derivative defined in [14], $\Xi:=[0,\mathcal{K}]$, $\Psi:\Xi\times C([-\zeta,\delta],\mathbb{R})\times\mathbb{R}\to\mathbb{R}$ is a given function, $\lambda\in C(\Xi\times\mathbb{R},\mathbb{R}\backslash\{0\})$, $\tilde{\xi}\in C([\mathcal{K},\mathcal{K}+\delta],\mathbb{R})$ and $\xi\in C([-\zeta,0],\mathbb{R})$ with $\xi(0)=0$. By y^t, we denote the element of $C([-\zeta,\delta])$ defined by:

$$y^t(\epsilon)=y(t+\epsilon),\ \ \epsilon\in[-\zeta,\delta].$$

As previously stated, there are several papers in which the authors explore various forms of differential problems using various factional operators such as Caputo, Atangana, Riemann-Liouville, Hadamard, Caputo-Fabrizio, Riesz derivative and Hilfer fractional derivatives, and many others. So, in this paper, we attempt to carry on the path of these investigations by introducing a new problem involving the improved conformable fractional derivative defined in [14] and in the context of hybrid equations. As far as we know, no such study exists in the literature.

This paper has the following structure: Section 2 presents certain notations and preliminaries about the improved conformable fractional derivatives used throughout this manuscript. In Section 3, we provide an existence result for problem (1.1)-(1.3), that is by using a fixed point theorem of Dhage [13], with mixed Lipschitz and Carathéodory conditions. Finally, in the last section, we give an example to illustrate the applicability of our main result.

2. Preliminaries

This section introduces the notations, definitions, and preliminary information that will be used throughout the paper. $C([-\zeta,\delta],\mathbb{R})$ denote the Banach space of all continuous functions from $[-\zeta,\delta]$ into $\mathbb{R}$ with the following norm

$$\|y\|_{[-\zeta,\delta]}=\sup_{-\zeta\le\tau\le\delta}\{|y(\tau)|\}.$$

Also, define the following space:

$$\mathcal{C}=\Bigg\{\alpha:[-\zeta,\mathcal{K}+\delta]\longmapsto\mathbb{R}:\alpha\mid_{[-\zeta,0]}\in C([-\zeta,0]),\alpha\mid_{[0,\mathcal{K}]}\in C^1([0,\mathcal{K}])$$
$$\text{and }\alpha\mid_{[\mathcal{K},\mathcal{K}+\delta]}\in C([\mathcal{K},\mathcal{K}+\delta])\Bigg\},$$

with the norm

$$\|\alpha\|_{\mathcal{C}}=\sup\{|\alpha(\tau)|:-\zeta\le\tau\le\mathcal{K}+\delta\}.$$

Consider the space $X_b^p(0,\mathcal{K})$, $(b\in\mathbb{R},\ 1\le p\le\infty)$ of those real-valued Lebesgue measurable functions Ψ on $[0,\mathcal{K}]$ for which $\|\Psi\|_{X_b^p}<\infty$, with:

$$\|\Psi\|_{X_b^p}=\left(\int_0^{\mathcal{K}}|\tau^b\Psi(\tau)|^p\frac{d\tau}{\tau}\right)^{\frac{1}{p}},\ \ (1\le p<\infty,b\in\mathbb{R}).$$

Definition 2.1. ([17]) The conformable fractional derivative of a given function $\psi:[0,+\infty)\longrightarrow\mathbb{R}$ of order ϑ is defined by

$$\mathcal{T}_\vartheta(\psi)(t)=\lim_{\varepsilon\to0}\frac{\psi\left(t+\varepsilon t^{1-\vartheta}\right)-\psi(t)}{\varepsilon}$$

for $t > 0$ and $\vartheta \in (0, 1]$. If ψ is ϑ-differentiable in some $(0, \mathcal{K})$, $\mathcal{K} > 0$, and $\lim_{t\to 0+} \mathcal{T}_{\mathcal{K}}(\psi)(t)$ exists, then define $\mathcal{T}_\vartheta(\psi)(0) = \lim_{t\to 0+} \mathcal{T}_\vartheta(\psi)(t)$. If the conformable fractional derivative of ψ of order ϑ exists, then we simply say that ψ is ϑ-differentiable. It is easy to see that if ψ is differentiable, then $\mathcal{T}_\vartheta(\psi)(t) = t^{1-\vartheta}\psi'(t)$.

Definition 2.2. (The improved Caputo-type conformable fractional derivative [14]) The improved Caputo-type conformable fractional derivative of order ϑ of a given function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by

$$ {}_{\mathcal{K}}^{C}\tilde{\mathcal{T}}_\vartheta(\psi)(\tau) = \lim_{\varepsilon\to 0}\left[(1-\vartheta)(\psi(\tau) - \psi(\mathcal{K})) + \vartheta\frac{\psi\left(\tau + \varepsilon(\tau-\mathcal{K})^{1-\vartheta}\right) - \psi(\tau)}{\varepsilon}\right], $$

where $-\infty < \mathcal{K} < \tau < +\infty, \mathcal{K}$ is a given number and $\vartheta \in [0, 1]$.

Definition 2.3. (The improved Riemann-Liouville-type conformable fractional derivative [14]) The improved Riemann-Liouville-type conformable fractional derivative of order ϑ of a given function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by

$$ {}_{\mathcal{K}}^{RL}\tilde{\mathcal{T}}_\vartheta(\psi)(\tau) = \lim_{\varepsilon\to 0}\left[(1-\vartheta)\psi(\tau) + \vartheta\frac{\psi\left(\tau + \varepsilon(\tau-\mathcal{K})^{1-\vartheta}\right) - \psi(\tau)}{\varepsilon}\right], $$

where $-\infty < \mathcal{K} < \tau < +\infty, \mathcal{K}$ is a given number and $\vartheta \in [0, 1]$.

Lemma 2.1. *([14]) If $\vartheta \in [0, 1]$, f and $\tilde{f}$ are two ϑ-differentiable functions at a point t and m, n are two given numbers, then the improved conformable fractional derivatives satisfy the following properties:*

- ${}_{a}^{C}\tilde{\mathcal{T}}_\vartheta(mf + n\tilde{f}) = m{}_{a}^{C}\tilde{\mathcal{T}}_\vartheta(f) + n{}_{a}^{C}\tilde{\mathcal{T}}_\vartheta(\tilde{f})$
- ${}_{a}^{RL}\tilde{\mathcal{T}}_\vartheta(mf + n\tilde{f}) = m{}_{a}^{RL}\tilde{\mathcal{T}}_\vartheta(f) + n{}_{a}^{RL}\tilde{\mathcal{T}}_\vartheta(\tilde{f})$
- ${}_{a}^{RL}\tilde{\mathcal{T}}_\vartheta(f\tilde{f}) = (1-\vartheta){}_{a}^{RL}\tilde{\mathcal{T}}_\vartheta(f)\tilde{f} + f{}_{a}^{RL}\tilde{\mathcal{T}}_\vartheta(\tilde{f}) - (1-\vartheta)f\tilde{f}$
- ${}_{a}^{RL}\tilde{\mathcal{T}}_\vartheta(f(\tilde{f}(t))) = (1-\vartheta)f(\tilde{f}(t)) + \vartheta f'(\tilde{f}(t))\mathcal{T}_\vartheta(\tilde{f}(t))$.

Definition 2.4. (The ϑ-fractional integral [14]) For $\vartheta \in (0, 1]$ and a continuous function f, let

$$ (\mathcal{I}_\vartheta f)(t) = \frac{1}{\vartheta}\int_0^t \frac{f(s)}{s^{1-\vartheta}} e^{\left(1-\vartheta/\vartheta^2\right)\left(s^\vartheta - t^\vartheta\right)} ds. $$

When $\vartheta = 1, \mathcal{I}_1(f) = \int_0^t f(s)ds$, the usual Riemann integral.

Lemma 2.2. *([14]) If $\vartheta \in [0, 1]$, ψ is ϑ-differentiable function at a point t and $\psi(0) = 0$, then we have:*

- $\left(\mathcal{I}_\vartheta {}_{0}^{C}\tilde{\mathcal{T}}_\vartheta(\psi)\right)(t) = {}_{0}^{C}\tilde{\mathcal{T}}_\vartheta\left(\mathcal{I}_\vartheta\psi\right)(t) = \psi(t)$;
- $\left(\mathcal{I}_\vartheta {}_{0}^{RL}\tilde{\mathcal{T}}_\vartheta(\psi)\right)(t) = {}_{0}^{RL}\tilde{\mathcal{T}}_\vartheta\left(\mathcal{I}_\vartheta\psi\right)(t) = \psi(t)$.

Lemma 2.3. *([13]) Let $\mathcal{D}$ be a closed, convex, bounded and nonempty subset of a Banach algebra $(E, \|\cdot\|)$, and let $\mathcal{S} : E \to E$ and $\tilde{\mathcal{S}} : \mathcal{D} \to E$ be two operators such that*

1): *$\mathcal{S}$ is Lipschitzian with Lipschitz constant η,*
2): *$\tilde{\mathcal{S}}$ is completely continuous,*
3): *$y = \mathcal{S}y\tilde{\mathcal{S}}z \Rightarrow y \in \mathcal{D}$ for all $z \in \mathcal{D}$*
4): *$\eta L < 1$, where $L = \|\tilde{\mathcal{S}}(\mathcal{D})\| = \sup\{\|\tilde{\mathcal{S}}(z)\| : z \in \mathcal{D}\}$.*

Then the operator equation $\mathcal{S}y\tilde{\mathcal{S}}y = y$ has a solution in $\mathcal{D}$.

3. Existence of Solutions

We need the following technical lemmas for the proof of the existence of our problem.

Lemma 3.4. *Let* $0 < \vartheta < 1$, $\tilde{\xi} \in C([\mathcal{K}, \mathcal{K}+\delta], \mathbb{R})$ *and* $\xi \in C([-\zeta, 0], \mathbb{R})$ *with* $\xi(0) = 0$, $\sigma \in C(\Xi, \mathbb{R}\backslash\{0\})$ *and* $\Lambda : \Xi \to \mathbb{R}$ *be a continuous function. Then problem*

$$ {}_0^C\tilde{\mathcal{T}}_\vartheta\left(\frac{y(t)}{\sigma(t)}\right) = \Lambda(t), \quad t \in (0, \mathcal{K}], \tag{3.4}$$

$$ y(t) = \xi(t), \; t \in [-\zeta, 0], \; \zeta > 0, \tag{3.5}$$

$$ y(t) = \tilde{\xi}(t), \; t \in [\mathcal{K}, \mathcal{K}+\delta], \; \delta > 0, \tag{3.6}$$

has the following solution

$$ y(t) = \begin{cases} \xi(t), \; \textit{if} \;\; t \in [-\zeta, 0], \\ \dfrac{\sigma(t)}{\vartheta}\displaystyle\int_0^t \frac{\Lambda(s)}{s^{1-\vartheta}} e^{\frac{(1-\vartheta)(s^\vartheta - t^\vartheta)}{\vartheta^2}} ds, \; \textit{if} \;\; t \in \Xi, \\ \tilde{\xi}(t), \; \textit{if} \;\; t \in [\mathcal{K}, \mathcal{K}+\delta]. \end{cases} \tag{3.7}$$

Proof. To obtain the integral equation (3.7), we apply the ϑ-fractional integral to (3.4), and by Lemma 2.2 we get

$$ y(t) = \frac{\sigma(t)}{\vartheta}\int_0^t \frac{\Lambda(s)}{s^{1-\vartheta}} e^{\frac{(1-\vartheta)(s^\vartheta - t^\vartheta)}{\vartheta^2}} ds. \tag{3.8}$$

Now, we apply the improved Caputo-type conformable fractional derivative of order ϑ to both sides of (3.8), for $t \in \Xi$ we obtain

$$ {}_0^C\tilde{\mathcal{T}}_\vartheta\left(\frac{y(t)}{\sigma(t)}\right) = \Lambda(t).$$

Also, it is clear that $y(0) = 0$. □

Lemma 3.5. *A function* $y \in \mathcal{C}$ *is a solution of to problem* (1.1)-(1.3) *if and only if it satisfies the following integral equation*

$$ y(t) = \begin{cases} \xi(t), \; \textit{if} \;\; t \in [-\zeta, 0], \\ \dfrac{\lambda(t, y(t))}{\vartheta}\displaystyle\int_0^t s^{\vartheta-1} e^{\frac{(1-\vartheta)(s^\vartheta - t^\vartheta)}{\vartheta^2}} \Psi\left(s, y^s(\cdot), {}_0^C\tilde{\mathcal{T}}_\vartheta\left(\frac{y(s)}{\lambda(s, y(s))}\right)\right) ds, \; \textit{if} \;\; t \in \Xi, \\ \tilde{\xi}(t), \; \textit{if} \;\; t \in [\mathcal{K}, \mathcal{K}+\delta]. \end{cases}$$

In the sequel, the following hypotheses are used:

(H_1): The function $\Psi : \Xi \times C([-\zeta, \delta], \mathbb{R}) \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous.

(H_2)**:** There exist functions $p, q_1, q_2 \in C(\Xi, [0, \infty))$ such that

$$ |\lambda(t, \bar{\beta_1}) - \lambda(t, \bar{\beta_2})| \le p(t)|\bar{\beta_1} - \bar{\beta_2}|$$

and

$$ |\Psi(t, \beta_1, \bar{\beta_1}) - \Psi(t, \beta_2, \bar{\beta_2})| \le q_1(t)\|\beta_1 - \beta_2\|_{[-\zeta,\delta]} + q_2(t)|\bar{\beta_1} - \bar{\beta_2}|,$$

for $t \in \Xi$, $\beta_1, \beta_2 \in C([-\zeta, \delta], \mathbb{R})$, $\bar{\beta_1}, \bar{\beta_2} \in \mathbb{R}$, with

$$p^* = \sup_{t\in\Xi} p(t),\ q_1^* = \sup_{t\in\Xi} q_1(t) \text{ and } q_2^* = \sup_{t\in\Xi} q_2(t) < 1.$$

(H_3)**:** There exists a number $R > 0$ such that

$$R \geq \max\left\{\|\xi\|_{[-\zeta,0]}, \|\tilde{\xi}\|_{[\mathcal{K},\mathcal{K}+\delta]}\right\},$$

and

$$R \geq \frac{\lambda^*}{1-\ell}\left[\frac{(\Psi^* + q_1^* R)\left(1 - e^{\frac{(\vartheta-1)\mathcal{K}^\vartheta}{\vartheta^2}}\right)}{(1-\vartheta)(1-q_2^*)}\right],$$

where $\lambda^* = \sup_{t\in\Xi} |\lambda(t,0)|$, $\Psi^* = \sup_{t\in\Xi} |\Psi(t,0,0)|$ and

$$\ell = p^*\left[\frac{(\Psi^* + q_1^* R)\left(1 - e^{\frac{(\vartheta-1)\mathcal{K}^\vartheta}{\vartheta^2}}\right)}{(1-\vartheta)(1-q_2^*)}\right].$$

Now we declare and demonstrate our existence result for problem (1.1)-(1.3) based on Lemma 2.3.

Theorem 3.1. *Assume that* (H_1)–(H_3) *hold. If*

$$\max\{p^* R, \ell\} < 1, \tag{3.9}$$

then problem (1.1)-(1.3) admit at least one solution defined on Ξ.

Proof. We define a subset Υ of $\mathcal{C}$ by

$$\Upsilon = \{y \in \mathcal{C} : \|y\|_{\mathcal{C}} \leq R\}.$$

We consider the operator $\Im : \mathcal{C} \longmapsto \mathcal{C}$ defined by

$$(\Im y)(t) = \begin{cases} \xi(t), \text{ if } \ t \in [-\zeta, 0], \\ \dfrac{\lambda(t,y(t))}{\vartheta} \displaystyle\int_0^t s^{\vartheta-1} e^{\frac{(1-\vartheta)(s^\vartheta - t^\vartheta)}{\vartheta^2}} \varrho(s)ds, \text{ if } \ t \in \Xi, \\ \tilde{\xi}(t), \text{ if } \ t \in [\mathcal{K}, \mathcal{K}+\delta], \end{cases} \tag{3.10}$$

and define two operators $T_1 : \mathcal{C} \to \mathcal{C}$ by

$$(T_1 y)(t) = \begin{cases} 1, \text{ if } \ t \in [-\zeta, 0], \\ \lambda(t, y(t)), \text{ if } \ t \in \Xi \\ 1, \text{ if } \ t \in [\mathcal{K}, \mathcal{K}+\delta], \end{cases} \tag{3.11}$$

and $T_2 : \Upsilon \to \mathcal{C}$ by

$$(T_2 y)(t) = \begin{cases} \xi(t), \text{ if } \ t \in [-\zeta, 0], \\ \dfrac{1}{\vartheta} \displaystyle\int_0^t s^{\vartheta-1} e^{\frac{(1-\vartheta)(s^\vartheta - t^\vartheta)}{\vartheta^2}} \varrho(s)ds, \quad t \in \Xi, \\ \tilde{\xi}(t), \text{ if } \ t \in [\mathcal{K}, \mathcal{K}+\delta], \end{cases} \tag{3.12}$$

where ϱ is a function satisfying the following functional equation

$$\varrho(t) = \Psi\left(t, x^t(\cdot), \varrho(t)\right).$$

Then we get $\Im y = T_1 y T_2 y$. We shall show that the operators T_1 and T_2 satisfies all the conditions of Lemma 2.3. The proof will be given in several steps.

Step 1: The operator T_1 is a Lipschitz on $\mathcal{C}$.
Let $x, y \in \mathcal{C}$ and $t \in [-\zeta, 0] \cup [\mathcal{K}, \mathcal{K}+\delta]$. Then

$$|(T_1x)(t) - (T_1y)(t)| = 0.$$

And for $t \in \Xi$, by (H_2) we have

$$\begin{aligned} |(T_1x)(t) - (T_1y)(t)| &\leq |\lambda(t, x(t)) - \lambda(t, y(t))|, \\ &\leq p(t)|x(t) - y(t)|, \\ &\leq p^*\|x - y\|_{\mathcal{C}}. \end{aligned}$$

Thus

$$\|T_1x - T_1y\|_{\mathcal{C}} \leq p^*\|x - y\|_{\mathcal{C}}.$$

Step 2: The operator T_2 is completely continuous on Υ.
Let $\{y_n\}$ be sequence in Υ such that $y_n \to y$ in Υ. If $t \in [-\zeta, 0]$ or $t \in [\mathcal{K}, \mathcal{K}+\delta]$ then

$$|(T_2y_n)(t) - (T_2y)(t)| = 0.$$

If $t \in \Xi$, we have

$$|(T_2y_n)(t) - (T_2y)(t)| \leq \frac{1}{\vartheta}\int_0^t s^{\vartheta-1} e^{\frac{(1-\vartheta)(s^\vartheta - t^\vartheta)}{\vartheta^2}} |\varrho_n(s) - \varrho(s)|\, ds,$$

where

$$\begin{aligned} \varrho_n(t) &= \Psi\left(t, y_n^t(\cdot), \varrho_n(t)\right), \\ \varrho(t) &= \Psi\left(t, y^t(\cdot), \varrho(t)\right). \end{aligned}$$

Since $y_n \to y$ and Ψ is continuous function then we get

$$\varrho_n(s) \to \varrho(s) \text{ as } n \to \infty.$$

Thus, by Lebesgue's Dominated convergence Theorem, we obtain

$$\|T_2y_n - T_2y\|_{\mathcal{C}} \to 0 \text{ as } n \to \infty.$$

Then T_2 is continuous. We will now prove that $T_2(\Upsilon)$ is uniformly bounded on $\mathcal{C}$. Let $y \in \Upsilon$.
For $t \in [-\zeta, 0]$, by (3.12) we have

$$|(T_2y)(t)| = |\xi(t)| \leq \|\xi\|_{[-\zeta,0]},$$

and if $t \in [\mathcal{K}, \mathcal{K}+\delta]$, we get

$$|(T_2y)(t)| = |\tilde{\xi}(t)| \leq \|\tilde{\xi}\|_{[\mathcal{K},\mathcal{K}+\delta]}.$$

And, for $t \in \Xi$ we have

$$|(T_2y)(t)| \leq \frac{1}{\vartheta}\int_0^t s^{\vartheta-1} e^{\frac{(1-\vartheta)(s^\vartheta - t^\vartheta)}{\vartheta^2}} \left|\Psi\left(s, y^s(\cdot), {}_0^C\tilde{\mathcal{T}}_\vartheta\left(\frac{y(s)}{\lambda(s, y(s))}\right)\right)\right| ds. \tag{3.13}$$

By the hypothesis (H_2), for $t \in \Xi$, we have

$$\begin{aligned} |\varrho(t)| &= \left|\Psi\left(t, y^t(\cdot), \varrho(t)\right) - \Psi(t, 0, 0) + \Psi(t, 0, 0)\right| \\ &\leq \left|\Psi\left(t, y^t(\cdot), \varrho(t)\right) - \Psi(t, 0, 0)\right| + \left|\Psi(t, 0, 0)\right| \end{aligned}$$

$$\leq q_1(t)\|y^t(\cdot)\|_{[-\zeta,\delta]} + q_2(t)|\varrho(t)| + \Psi^*$$
$$\leq q_1^*\|y^t(\cdot)\|_{[-\zeta,\delta]} + q_2^*|\varrho(t)| + \Psi^*,$$

then

$$|\varrho(t)| \leq \frac{\Psi^* + q_1^* R}{1 - q_2^*}.$$

Thus, from (3.13) we get

$$\begin{aligned}|(T_2y)(t)| &\leq \frac{(\Psi^* + q_1^*R)\left(1 - e^{\frac{(\vartheta-1)t^\vartheta}{\vartheta^2}}\right)}{(1-\vartheta)(1-q_2^*)} \\ &\leq \frac{(\Psi^* + q_1^*R)\left(1 - e^{\frac{(\vartheta-1)\mathcal{K}^\vartheta}{\vartheta^2}}\right)}{(1-\vartheta)(1-q_2^*)}.\end{aligned}$$

Then, we conclude that

$$\|T_2y\|_{\mathcal{C}} \leq \max\left\{\frac{(\Psi^* + q_1^*R)\left(1 - e^{\frac{(\vartheta-1)\mathcal{K}^\vartheta}{\vartheta^2}}\right)}{(1-\vartheta)(1-q_2^*)}, \|\xi\|_{[-\zeta,0]}, \|\tilde{\xi}\|_{[\mathcal{K},\mathcal{K}+\delta]}\right\}.$$

We can deduce now that the operator T_2 is uniformly bounded on Υ. Following that, we prove that the operator $T_2\Upsilon$ equicontinuous. We take $y \in \Upsilon$ and let $\gamma_1, \gamma_2 \in \Xi = [0, \mathcal{K}], \gamma_1 < \gamma_2$, then

$$\begin{aligned}|(T_2y)(\gamma_2) - (T_2y)(\gamma_1)| &\leq \left|\frac{1}{\vartheta}\int_0^{\gamma_2} s^{\vartheta-1} e^{\frac{(1-\vartheta)(s^\vartheta - \gamma_2^\vartheta)}{\vartheta^2}} \varrho(s)ds - \frac{1}{\vartheta}\int_0^{\gamma_1} s^{\vartheta-1} e^{\frac{(1-\vartheta)(s^\vartheta-\gamma_1^\vartheta)}{\vartheta^2}} \varrho(s)ds\right| \\ &\leq \frac{\Psi^* + q_1^*R}{(1-\vartheta)(1-q_2^*)}\left[2 - 2e^{\frac{(1-\vartheta)(\gamma_1^\vartheta - \gamma_2^\vartheta)}{\vartheta^2}} + e^{\frac{(\vartheta-1)\gamma_1^\vartheta}{\vartheta^2}} - e^{\frac{(\vartheta-1)\gamma_2^\vartheta}{\vartheta^2}}\right].\end{aligned}$$

As $\gamma_1 \longrightarrow \gamma_2$ the right hand side of the above inequality tends to zero. This proves that $T_2\Upsilon$ is equicontinuous on Ξ. The equicontinuity for the other cases is obvious, thus we omit the details. Consequently, we conclude that T_2 is completely continuous on Υ using Arzela-Ascoli Theorem.

Step 3: Now we show that the third hypothesis of Lemma 2.3 is satisfied. Let $x \in \mathcal{C}$ and $y \in \Upsilon$ be arbitrary such that $x = T_1xT_2y$. Then, for $t \in [-\zeta, 0]$ and by hypothesis (H_3), we have

$$|x(t)| = |(T_1xT_2y)(t)| = |\xi(t)| \leq \|\xi\|_{[-\zeta,0]} \leq R,$$

and for $t \in [\mathcal{K}, \mathcal{K}+\delta]$, we get

$$|x(t)| = |(T_1xT_2y)(t)| = |\tilde{\xi}(t)| \leq \|\tilde{\xi}\|_{[\mathcal{K},\mathcal{K}+\delta]} \leq R.$$

And, for $t \in \Xi$, we obtain

$$\begin{aligned}
|x(t)| &= |(T_1xT_2y)(t)| \\
&= |(T_1x)(t)|\,|(T_2y)(t)| \\
&= \frac{|\lambda(t,x(t))|}{\vartheta}\int_0^t s^{\vartheta-1}e^{\frac{(1-\vartheta)(s^\vartheta-t^\vartheta)}{\vartheta^2}}|\varrho(s)|ds \\
&\leq (|\lambda(t,x(t))-\lambda(t,0)|+|\lambda(t,0)|)\left[\frac{(\Psi^*+q_1^*R)\left(1-e^{\frac{(\vartheta-1)\mathcal{K}^\vartheta}{\vartheta^2}}\right)}{(1-\vartheta)(1-q_2^*)}\right] \\
&\leq (p^*\|x\|_{\mathcal{C}}+\lambda^*)\left[\frac{(\Psi^*+q_1^*R)\left(1-e^{\frac{(\vartheta-1)\mathcal{K}^\vartheta}{\vartheta^2}}\right)}{(1-\vartheta)(1-q_2^*)}\right],
\end{aligned}$$

then,

$$\begin{aligned}
|x(t)| &\leq \frac{\lambda^*\left[\dfrac{(\Psi^*+q_1^*R)\left(1-e^{\frac{(\vartheta-1)\mathcal{K}^\vartheta}{\vartheta^2}}\right)}{(1-\vartheta)(1-q_2^*)}\right]}{1-p^*\left[\dfrac{(\Psi^*+q_1^*R)\left(1-e^{\frac{(\vartheta-1)\mathcal{K}^\vartheta}{\vartheta^2}}\right)}{(1-\vartheta)(1-q_2^*)}\right]} \\
&\leq R.
\end{aligned}$$

Thus,

$$\|x\|_{\mathcal{C}} \leq R.$$

As consequence, we have $x \in \Upsilon$.

Step 4: Now, we show that $p^*L < 1$, where $L = \|T_2(\Upsilon)\|_{\mathcal{C}} = \sup\{\|T_2y\|_{\mathcal{C}} : y \in \Upsilon\}$. Since

$$L \leq \max\left\{\frac{(\Psi^*+q_1^*R)\left(1-e^{\frac{(\vartheta-1)\mathcal{K}^\vartheta}{\vartheta^2}}\right)}{(1-\vartheta)(1-q_2^*)}, \|\xi\|_{[-\zeta,0]}, \|\tilde{\xi}\|_{[\mathcal{K},\mathcal{K}+\delta]}\right\},$$

then

$$p^*L \leq \max\{\ell, p^*R\} < 1.$$

Thus, we have proven the last hypothesis of Lemma 2.3. Then, as a consequence of steps 1 to 4 with Theorem 3.1, we deduce that the operator equation $\Im y = T_1yT_2y = y$ has at least one solution which is a fixed point for $\Im$. This fixed point is a solution to problem (1.1)-(1.3). □

4. EXAMPLE

Example 4.1. Consider the following initial value problem with the improved Caputo-type conformable fractional derivative:

$$\begin{cases} y(t) = \cos(t), \quad t \in [\pi, 2\pi], \\ {}_0^C\tilde{\mathcal{T}}_{\frac{1}{2}}\left(\dfrac{y(t)}{\lambda(t, y(t))}\right) = \Psi\left(t, y^t(\cdot), {}_0^C\tilde{\mathcal{T}}_{\frac{1}{2}}\left(\dfrac{y(t)}{\lambda(t, y(t))}\right)\right), \quad t \in (0, \pi], \\ y(t) = \sin(t), \quad t \in [-\pi, 0], \end{cases} \tag{4.14}$$

where $\Xi = [0, \pi]$, $\vartheta = \frac{1}{2}$, $\zeta = \delta = \pi$ and

$$\lambda(t, y(t)) = \frac{1}{103e^t}\left(|\sin(t)y(t)| + |\cos(t)| + 1\right), \ t \in \Xi, \ y \in \mathcal{C}.$$

Set

$$\Psi(t, y, \bar{y}) = \frac{|sin(t)||y| + |\bar{y}|}{425(1 + |y| + |\bar{y}|)}, \ t \in \Xi, \ y \in C([-\zeta, \delta], \mathbb{R}) \text{ and } \bar{y} \in \mathbb{R}.$$

For each $y, \bar{y} \in \mathbb{R}$ and $t \in \Xi$, we have

$$|\lambda(t, y) - \lambda(t, \bar{y})| \leq \frac{|\sin(t)|}{103e^t}|y - \bar{y}|,$$

and for each $y_1, \bar{y}_1 \in \mathbb{R}$, $y_2, \bar{y}_2 \in C([-\zeta, \delta], \mathbb{R})$ and $t \in \Xi$, we have

$$|\Psi(t, y_2, y_1) - \Psi(t, \bar{y}_2, \bar{y}_1)| \leq \frac{|\sin(t)|}{425}\|y_2 - \bar{y}_2\|_{[-\zeta,\delta]} + \frac{1}{425}|y_1 - \bar{y}_1|.$$

Hence condition (H_2) is satisfied with

$$p(t) = \frac{|\sin(t)|}{103e^t}, p^* = \frac{1}{103},$$

and

$$q_1(t) = \frac{|\sin(t)|}{425}, q_2(t) = \frac{1}{425}, q_1^* = q_2^* = \frac{1}{425}.$$

Also, the hypothesis (H_3) and the condition (3.9) of Theorem 3.1 are satisfied if we take

$$1 \leq R < 103.$$

Then the problem (4.14) has at least one solution defined on $[-\pi, 2\pi]$.

DECLARATIONS

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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