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On a Subclass of Meromorphic Functions Defined by a Differential Operator on Hilbert Space

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ABSTRACT. In this article, we introduce and study a new subclass of meromorphic functions associated with a differential operator on Hilbert space. Coefficient estimates, growth and distortion bounds, extreme points, radii results, convex linear combinations, Hadamard product and integral transforms are obtained.

1. INTRODUCTION

Denote by Σ_0 the class of meromorphic functions f of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in $\mathscr{D} = \{z \in \mathbb{C} : 0 < |z| < 1\}$. The Hadamard product (or convolution) [5] of functions $f \in \Sigma_0$ given by (1.1) and $g \in \Sigma_0$ of the form

$$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n \tag{1.2}$$

is defined by

$$(f * g)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n,\ z \in \mathscr{D}. \tag{1.3}$$

The function $f \in \Sigma_0$ is said to be meromorphically starlike and meromorphically convex of order $\rho,\ 0 \le \rho < 1$ if it satisfies the following conditions :

$$\Re e \left\{ -\left(\frac{zf'(z)}{f(z)} \right) \right\} > \rho \text{ and } \Re e \left\{ -\left(1 + \frac{zf''(z)}{f'(z)} \right) \right\} > \rho,\ z \in \mathscr{D},\ \text{respectively.}$$

The classes of meromorphically starlike and meromorphically convex functions of order ρ are denoted by $\mathscr{S}^*(\rho)$ and $\mathscr{K}(\rho)$. Moreover the function $f \in \Sigma_0$ is said to be meromorphically close to convex of order β if there exist a function $g \in \mathscr{S}^*$ such that

$$\Re e \left\{ -\left(\frac{zf'(z)}{g(z)} \right) \right\} > \beta,\ (0 \le \rho < 1,\ 0 \le \beta < 1),\ z \in \mathscr{D}.$$

The class of meromorphically close to convex functions of order β is denoted by $\mathscr{K}_0(\beta, \rho)$.

Let $\mathcal{H}$ be a complex Hilbert space and $\mathbb{T}$ be a bounded linear transformation on $\mathcal{H}$. For a complex analytic function f in a domain E of the complex plane containing the spectrum

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$\sigma(\mathbb{T})$ of the bounded linear operator $\mathbb{T}$, let $f(\mathbb{T})$ denote the operator on $\mathcal{H}$ defined by the Riesz-Dunford integral [4]

$$f(\mathbb{T}) = \frac{1}{2\pi i}\int_C (z\mathbb{I} - \mathbb{T})^{-1} f(z)dz$$

where $\mathbb{I}$ is the identity operator on $\mathcal{H}$ and C is a positively-oriented simple closed rectifiable contour containing the spectrum $\sigma(\mathbb{T})$ in the interior domain [6]. The operator $f(\mathbb{T})$ can also be defined by the following series [7] :

$$f(\mathbb{T}) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\mathbb{T}^n \tag{1.4}$$

which converges in the norm topology.

For $f \in \Sigma_0$, the differential operator introduced by Deniz and Ozkan [3] was defined as

$$\begin{aligned}
&\mathfrak{D}_\lambda^0 f(z) = f(z)\\
&\mathfrak{D}_\lambda^1 f(z) = \mathfrak{D}_\lambda f(z) = \lambda z^3 (f(z))''' + (2\lambda + 1)z^2(f(z))'' + zf'(z)\\
&\mathfrak{D}_\lambda^2 f(z) = \mathfrak{D}_\lambda(\mathfrak{D}_\lambda^1 f(z))\\
&\qquad\qquad .\\
&\qquad\qquad .\\
&\qquad\qquad .\\
&\mathfrak{D}_\lambda^m f(z) = \mathfrak{D}_\lambda(\mathfrak{D}_\lambda^{m-1} f(z))
\end{aligned}$$

where $\lambda \geq 0$ and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
For f given by (1.1), and from the definition of the operator $\mathfrak{D}_\lambda^m f(z)$, we have

$$\mathfrak{D}_\lambda^m f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \Psi^m(\lambda, n) a_n z^n \tag{1.5}$$

where

$$\Psi^m(\lambda, n) = n^{2m}[\lambda(n-1) + 1]^m. \tag{1.6}$$

Definition 1.1. A function f of the form (1.1) is said to be in the class $\mathcal{T}_m(\alpha, \mu, \lambda, \mathbb{T})$ if

$$\begin{aligned}
&\|\mathbb{T}(\mathfrak{D}_\lambda^m f(\mathbb{T})' - \{(\mu - 1)\mathfrak{D}_\lambda^m f(\mathbb{T}) + \mu\mathbb{T}(\mathfrak{D}_\lambda^m f(\mathbb{T}))'\}\| <\\
&\qquad \|\mathbb{T}(\mathfrak{D}_\lambda^m f(\mathbb{T}))' + (1 - 2\alpha)\{(\mu - 1)\mathfrak{D}_\lambda^m f(\mathbb{T}) + \mu\mathbb{T}(\mathfrak{D}_\lambda^m f(\mathbb{T}))'\}\|
\end{aligned} \tag{1.7}$$

for $0 \leq \alpha < 1,\ 0 \leq \mu < 1$, and all operators $\mathbb{T}$ with $\|\mathbb{T}\| < 1$ and $\|\mathbb{T}\| \neq \mathbb{O}$, where $\mathbb{O}$ denotes the zero operator on $\mathcal{H}$.

Many authors [8, 10, 12, 13] have defined and studied subclasses of analytic and meromorphic functions on the unit disk using Hilbert space operators. A new subclass of analytic univalent functions using subordination was studied by Srivastava $et.al.$ [15]. Generalization of results in this direction with the introduction of various operators were carried out by various researchers [1, 2, 11, 14, 16]. This resulted in the introduction of a subclass of meromorphic functions $\mathcal{T}_m(\alpha, \mu, \lambda, \mathbb{T})$ defined using Hilbert space operator and have obtained the necessary and sufficient condition for the functions to belong to this class, the distortion theorem, radii results, integral transforms and Hadamard product for the functions in this class are examined. This class appears to be of significant with the introduction of the Deniz and Ozkan operator to study the various results examined.

2. Coefficient bounds

Theorem 2.1. *A function f of the form* (1.1) *belongs to the class* $\mathcal{T}_m(\alpha,\mu,\lambda,\mathbb{T})$ *if and only if*

$$\sum_{n=1}^{\infty}[n+\alpha-\alpha\mu(n+1)]\Psi^m(\lambda,n)a_n \le 1-\alpha. \tag{2.8}$$

The result is sharp for the function

$$f(z)=\frac{1}{z}+\frac{1-\alpha}{[n+\alpha-\alpha\mu(n+1)]\Psi^m(\lambda,n)}z^n. \tag{2.9}$$

Proof. Let $f(\mathbb{T})=\mathbb{T}^{-1}+\sum_{n=1}^{\infty}a_n\mathbb{T}^n$. Assume that (2.8) holds. Then

$$\|\mathbb{T}(\mathfrak{D}_\lambda^m f(\mathbb{T})'-\{(\mu-1)\mathfrak{D}_\lambda^m f(\mathbb{T})+\mu\mathbb{T}(\mathfrak{D}_\lambda^m f(\mathbb{T}))'\}\|<$$
$$\|\mathbb{T}(\mathfrak{D}_\lambda^m f(\mathbb{T}))'+(1-2\alpha)\{(\mu-1)\mathfrak{D}_\lambda^m f(\mathbb{T})+\mu\mathbb{T}(\mathfrak{D}_\lambda^m f(\mathbb{T}))'\}\|$$

$$=\left\|\sum_{n=1}^{\infty}[(n+1)-\mu(n+1)]\Psi^m(\lambda,n)a_n\mathbb{T}^n\right\|-\Big\|2\mathbb{T}^{-1}(1-\alpha)-$$
$$\sum_{n=1}^{\infty}[n+(1-2\alpha)(\mu-1)+\mu(1-2\alpha)n]\Psi^m(\lambda,n)a_n\mathbb{T}^n\Big\|$$

$$\le\sum_{n=1}^{\infty}[n+\alpha-\alpha\mu(n+1)]\Psi^m(\lambda,n)a_n-(1-\alpha)\le 0,\ \text{by (2.8)},$$

Hence $f\in\mathcal{T}_m(\alpha,\mu,\lambda,\mathbb{T})$.

Conversely, let

$$\|\mathbb{T}(\mathfrak{D}_\lambda^m f(\mathbb{T})'-\{(\mu-1)\mathfrak{D}_\lambda^m f(\mathbb{T})+\mu\mathbb{T}(\mathfrak{D}_\lambda^m f(\mathbb{T}))'\}\|<$$
$$\|\mathbb{T}(\mathfrak{D}_\lambda^m f(\mathbb{T}))'+(1-2\alpha)\{(\mu-1)\mathfrak{D}_\lambda^m f(\mathbb{T})+\mu\mathbb{T}(\mathfrak{D}_\lambda^m f(\mathbb{T}))'\}\|.$$

$$\Rightarrow\left\|\sum_{n=1}^{\infty}[(n+1)(1-\mu)]\Psi^m(\lambda,n)a_n\mathbb{T}^{n+1}\right\|$$
$$\le\left\|2(1-\alpha)-\sum_{n=1}^{\infty}[n+(1-2\alpha)(\mu-1)+\mu(1-2\alpha)n]\Psi^m(\lambda,n)a_n\mathbb{T}^{n+1}\right\|.$$

Choosing $\mathbb{T}=e\mathbb{I}$, $(0<e<1)$ we get

$$\frac{\sum_{n=1}^{\infty}[(n+1)(1-\mu)]\Psi^m(\lambda,n)a_ne^{n+1}}{2(1-\alpha)-\sum_{n=1}^{\infty}[n+(1-2\alpha)(\mu-1)+\mu(1-2\alpha)n]\Psi^m(\lambda,n)a_ne^{n+1}}<1.$$

Letting $e\to 1$ in the above inequality, we obtain (2.8). □

Corollary 2.1. *If f of the form* (1.1) *is in the class* $\mathcal{T}_m(\alpha,\mu,\lambda,\mathbb{T})$ *then*

$$a_n\le\frac{1-\alpha}{[n+\alpha-\alpha\mu(n+1)]\Psi^m(\lambda,n)},\ (n\ge 1)$$

The result is sharp for the function

$$f(z)=\frac{1}{z}+\frac{1-\alpha}{[n+\alpha-\alpha\mu(n+1)]\Psi^m(\lambda,n)}z^n.$$

3. Growth and Distortion bounds

Theorem 3.2. *If f of the form* (1.1) *is in the class* $\mathcal{T}_m(\alpha,\mu,\lambda,\mathbb{T})$ *then*

$$\|f(\mathbb{T})\| \geq \frac{1}{\|\mathbb{T}\|} - \frac{1-\alpha}{(1+\alpha-2\alpha\mu)\Psi^m(\lambda,1)}\|\mathbb{T}\|$$

and

$$\|f(\mathbb{T})\| \leq \frac{1}{\|\mathbb{T}\|} + \frac{1-\alpha}{(1+\alpha-2\alpha\mu)\Psi^m(\lambda,1)}\|\mathbb{T}\|.$$

The result is sharp for

$$f(z) = \frac{1}{z} + \frac{1-\alpha}{[1+\alpha-2\alpha\mu]\Psi^m(\lambda,1)}z.$$

Proof. By Theorem(2.1), we have

$$(1+\alpha-2\alpha\mu)\Psi^m(\lambda,1)\sum_{n=2}^{\infty} a_n \leq \sum_{n=1}^{\infty}(n+\alpha-\alpha\mu(n+1))\Psi^m(\lambda,n)a_n \leq 1-\alpha.$$

Therefore

$$\sum_{n=1}^{\infty} a_n \leq \frac{1-\alpha}{(1+\alpha-2\alpha\mu)\Psi^m(\lambda,1)} \tag{3.10}$$

Also, $f(\mathbb{T}) = \mathbb{T}^{-1} + \sum_{n=1}^{\infty} a_n\mathbb{T}^n$, then

$$\frac{1}{\|\mathbb{T}\|} - \sum_{n=1}^{\infty} a_n\|\mathbb{T}\|^n \leq \|f(\mathbb{T})\| \leq \frac{1}{\|\mathbb{T}\|} + \sum_{n=1}^{\infty} a_n\|\mathbb{T}\|^n. \tag{3.11}$$

Since $\|\mathbb{T}\| < 1$, the above inequality becomes

$$\frac{1}{\|\mathbb{T}\|} - \|\mathbb{T}\|\sum_{n=1}^{\infty} a_n \leq \|f(\mathbb{T})\| \leq \frac{1}{\|\mathbb{T}\|} + \|\mathbb{T}\|\sum_{n=1}^{\infty} a_n. \tag{3.12}$$

Using (3.10) we get the result. □

Theorem 3.3. *If f of the form* (1.1) *is in the class* $\mathcal{T}_m(\alpha,\mu,\lambda,\mathbb{T})$ *then*

$$\|f'(\mathbb{T})\| \geq \frac{1}{\|\mathbb{T}\|^2} - \frac{1-\alpha}{(1+\alpha-2\alpha\mu)\Psi^m(\lambda,1)}$$

and

$$\|f'(\mathbb{T})\| \leq \frac{1}{\|\mathbb{T}\|^2} + \frac{1-\alpha}{(1+\alpha-2\alpha\mu)\Psi^m(\lambda,1)}.$$

The result is sharp for

$$f(z) = \frac{1}{z} + \frac{1-\alpha}{[1+\alpha-2\alpha\mu]\Psi^m(\lambda,1)}z.$$

4. Extreme Points

Theorem 4.4. *Let* $f_0(z) = \dfrac{1}{z}$ *and*

$$f_n(z) = \frac{1}{z} + \frac{1-\alpha}{[n+\alpha-\alpha\mu(n+1)]\Psi^m(\lambda,n)}z^n,\ (n \geq 1)$$

$0 \leq \alpha < 1,\ 0 \leq \mu < 1$. *Then* $f \in \mathcal{T}_m(\alpha,\mu,\lambda,\mathbb{T})$ *if and only if it can be expressed as*

$f(z) = \sum_{n=0}^{\infty} \mu_n f_n(z)$ *where* $\mu_n \geq 0, (n = 0, 1, 2, \dots)$ *and* $\sum_{n=0}^{\infty} \mu_n = 1$.

Proof. Assume that

$$f(z)=\sum_{n=0}^{\infty}\mu_n f_n(z)=\frac{1}{z}+\sum_{n=1}^{\infty}\mu_n\left[\frac{(1-\alpha)}{(n+\alpha-\alpha\mu(n+1))\Psi^m(\lambda,n)}\right]z^n.$$

Then

$$\sum_{n=1}^{\infty}[n+\alpha-\alpha\mu(n+1)]\Psi^m(\lambda,n)\times\frac{\mu_n(1-\alpha)}{(n+\alpha-\alpha\mu(n+1))\Psi^m(\lambda,n)}$$
$$=\sum_{n=1}^{\infty}\mu_n=1-\mu_0\leq 1.$$

By Theorem (2.1), $f\in\mathcal{T}_m(\alpha,\mu,\lambda,\mathbb{T})$.

Conversely assume that f is in the class $\mathcal{T}_m(\alpha,\mu,\lambda,\mathbb{T})$ then by Corollary (2.1),

$$a_n\leq\frac{1-\alpha}{(n+\alpha-\alpha\mu(n+1))\Psi^m(\lambda,n)}.$$

Set

$$\mu_n=\frac{(n+\alpha-\alpha\mu(n+1))\Psi^m(\lambda,n)}{1-\alpha}a_n,\ n=1,2,...$$

and $\mu_0=1-\sum_{n=1}^{\infty}\mu_n$. Then $f(z)=\sum_{n=0}^{\infty}\mu_n f_n(z)\in\mathcal{T}_m(\alpha,\mu,\lambda,\mathbb{T})$. □

5. Radii Results

Theorem 5.5. *Let $f\in\mathcal{T}_m(\alpha,\mu,\lambda,\mathbb{T})$. Then f is meromorphically close-to-convex of order δ, $(0\leq\delta<1)$ in the disc $|z|<r_1$, where*

$$r_1:=inf_{n\geq 1}\left[\frac{(1-\delta)[n+\alpha-\alpha\mu(n+1)]\Psi^m(\lambda,n)}{n(1-\alpha)}\right]^{\frac{1}{n+1}}.$$

The result is sharp for the extremal function given by (2.9).

Proof. It suffices to show that

$$\|f'(\mathbb{T})\mathbb{T}^2+1\|<1-\delta. \tag{5.13}$$

By Theorem (2.1),

$$\sum_{n=1}^{\infty}\frac{[n+\alpha-\alpha\mu(n+1)]\Psi^m(\lambda,n)}{1-\alpha}a_n\leq 1.$$

The inequality

$$\|f'(\mathbb{T})\mathbb{T}^2+1\|=\left\|\sum_{n=1}^{\infty}na_n\mathbb{T}^{n+1}\right\|\leq\sum_{n=1}^{\infty}na_n\|\mathbb{T}\|^{n+1}<1-\delta.$$

holds true if

$$\frac{n\|\mathbb{T}\|^{n+1}}{1-\delta}\leq\frac{[n+\alpha-\alpha\mu(n+1)]\Psi^m(\lambda,n)}{1-\alpha}$$

Then

$$\|\mathbb{T}\|^{n+1}\leq\frac{(1-\delta)[n+\alpha-\alpha\mu(n+1)]\Psi^m(\lambda,n)}{n(1-\alpha)},n\geq 1$$

which yields the close-to-convexity of the function and completes the proof. □

Theorem 5.6. *Let $f \in \mathcal{T}_m(\alpha, \mu, \lambda, \mathbb{T})$. Then f is meromorphically starlike of order $\delta, (0 \leq \delta < 1)$ in the disc $|z| < r_2$, where*

$$r_2 := inf_{n \geq 1} \left[\frac{(1-\delta)[n+\alpha-\alpha\mu(n+1)]\Psi^m(\lambda, n)}{(1-\alpha)(n+2-\delta)} \right]^{\frac{1}{n+1}}.$$

The result is sharp for the extremal function (2.9).

Proof. Let $f(\mathbb{T}) = \mathbb{T}^{-1} + \sum_{n=1}^{\infty} a_n \mathbb{T}^n$. Since f in $\mathcal{T}_m(\alpha, \mu, \lambda, \mathbb{T})$ is meromorphically starlike of order δ

$$\left\| -\frac{\mathbb{T} f'(\mathbb{T})}{f(\mathbb{T})} - 1 \right\| \leq 1 - \delta. \tag{5.14}$$

Substituting for f, in the above inequality,

$$\sum_{n=1}^{\infty} \left(\frac{n+2-\delta}{1-\delta} \right) \|\mathbb{T}\|^{n+1} a_n \leq 1. \tag{5.15}$$

By Theorem(2.1),

$$\sum_{n=1}^{\infty} \frac{[n+\alpha-\alpha\mu(n+1)]\Psi^m(\lambda, n)}{1-\alpha} a_n \leq 1.$$

Hence (5.15) holds true if

$$\left(\frac{n+2-\delta}{1-\delta} \right) \|\mathbb{T}\|^{n+1} \leq \frac{[n+\alpha-\alpha\mu(n+1)]\Psi^m(\lambda, n)}{1-\alpha}.$$

That is

$$\|\mathbb{T}\| \leq \left[\frac{(1-\delta)[n+\alpha-\alpha\mu(n+1)]\Psi^m(\lambda, n)}{(1-\alpha)(n+2-\delta)} \right]^{\frac{1}{n+1}}.$$

□

Theorem 5.7. *Let $f \in \mathcal{T}_m(\alpha, \mu, \lambda, \mathbb{T})$. Then f is meromorphically convex of order $\delta, (0 \leq \delta < 1)$ in the disc $|z| < r_3$, where*

$$r_3 := inf_{n \geq 1} \left[\frac{(1-\delta)[n+\alpha-\alpha\mu(n+1)]\Psi^m(\lambda, n)}{(1-\alpha)n(n+2-\delta)} \right]^{\frac{1}{n+1}}.$$

The result is sharp for the extremal function (2.9).

Proof. The proof of the result is akin to Theorem (5.5), hence omitted. □

6. Convex combination

Theorem 6.8. *The class $\mathcal{T}_m(\alpha, \mu, \lambda, \mathbb{T})$ is closed under convex combination.*

Proof. Let $f(\mathbb{T}) = \frac{1}{\mathbb{T}} + \sum_{n=0}^{\infty} a_n \mathbb{T}^n$ and $g(\mathbb{T}) = \frac{1}{\mathbb{T}} + \sum_{n=0}^{\infty} b_n \mathbb{T}^n$ be in the class $\mathcal{T}_m(\alpha, \mu, \lambda, \mathbb{T})$,

then by Theorem (2.1),

$$\sum_{n=1}^{\infty} [n+\alpha-\alpha\mu(n+1)]\Psi^m(\lambda, n) a_n \leq 1 - \alpha.$$

and

$$\sum_{n=1}^{\infty} [n+\alpha-\alpha\mu(n+1)]\Psi^m(\lambda, n) b_n \leq 1 - \alpha.$$

For $0 \leq \eta \leq 1$, we define the function h as $h(\mathbb{T}) = \eta f(\mathbb{T}) + (1-\eta) g(\mathbb{T})$, then

$h(\mathbb{T}) = \mathbb{T}^{-1} + \sum_{n=1}^{\infty}[\eta a_n + (1-\eta)b_n]\mathbb{T}^n.$

Now we obtain

$$\begin{aligned}&\sum_{n=1}^{\infty}[n+\alpha-\alpha\mu(n+1)]\Psi^m(\lambda,n)[\eta a_n + (1-\eta)b_n]\\&= \eta\sum_{n=1}^{\infty}[n+\alpha-\alpha\mu(n+1)]\Psi^m(\lambda,n)a_n + (1-\eta)\sum_{n=1}^{\infty}[n+\alpha-\alpha\mu(n+1)]\Psi^m(\lambda,n)b_n\\&\leq \eta(1-\alpha) + (1-\eta)(1-\alpha) = 1-\alpha.\end{aligned}$$

Hence $h \in \mathcal{T}_m(\alpha,\mu,\lambda,\mathbb{T})$. □

7. Integral transforms

We examine integral transforms of functions in the class $\mathcal{T}_m(\alpha,\mu,\lambda,\mathbb{T})$ of the type considered by Goel and Sohi [9].

Theorem 7.9. *Let the function f given by* (1.1) *be in the class $\mathcal{T}_m(\alpha,\mu,\lambda,\mathbb{T})$, then*

$$F(z) = \kappa\int_0^1 u^{\kappa} f(uz)du,\ 0 < u \leq 1, 0 < \kappa < \infty.$$

is in $\mathcal{T}_m(I,\mu,\lambda,\mathbb{T})$ when

$$I = 1 - \frac{(1-\alpha)(1+2\mu)+\kappa}{(1+\alpha-2\alpha\mu)(\kappa+2)+(1-\alpha)(1-2\mu)\kappa}.$$

The result is sharp for the function

$$f(z) = \frac{1}{z} + \frac{(1-\alpha)}{(1+\alpha-2\alpha\mu)\Psi^m(\lambda,1)}z.$$

Proof. Let $f \in \mathcal{T}_m(\alpha,\mu,\lambda,\mathbb{T})$. Then

$$F(z) = \kappa\int_0^1 u^{\kappa} f(uz)du = \frac{1}{z} + \sum_{n=1}^{\infty}\frac{\kappa}{\kappa+n+1}a_n z^n.$$

We show that

$$\sum_{n=1}^{\infty}\left[\frac{\kappa(n+I-I\mu(n+1))\Psi^m(\lambda,n)}{(1-I)(\kappa+n+1)}\right]a_n \leq 1. \tag{7.16}$$

Since $f \in \mathcal{T}_m(\alpha,\mu,\lambda,\mathbb{T})$, we have

$$\sum_{n=1}^{\infty}\left[\frac{(n+\alpha-\alpha\mu(n+1))\Psi^m(\lambda,n)}{(1-\alpha)}\right]a_n \leq 1.$$

The inequality (7.16) satisfies if

$$\frac{\kappa(n+I-I\mu(n+1))}{(1-I)(\kappa+n+1)} \leq \frac{(n+\alpha-\alpha\mu(n+1))}{(1-\alpha)}.$$

we get

$$\begin{aligned}I &\leq \frac{[n+\alpha-\alpha\mu(n+1)](\kappa+n+1)-(1-\alpha)\kappa n}{[n+\alpha-\alpha\mu(n+1)](\kappa+n+1)+\kappa(1-\alpha)(1-\mu(n+1))}\\&= 1 - \frac{(1-\alpha)(1+\mu(n+1)+\kappa n)}{[n+\alpha-\alpha\mu(n+1)](\kappa+n+1)+\kappa(1-\alpha)(1-\mu(n+1))}.\end{aligned}$$

We show that the function

$$\varphi(n) = 1 - \frac{(1-\alpha)(1+\mu(n+1)+\kappa n)}{[n+\alpha-\alpha\mu(n+1)](\kappa+n+1)+\kappa(1-\alpha)(1-\mu(n+1))}$$

is an increasing function of n $(n \geq 1)$ and $\varphi(n) \geq \varphi(1)$. The desired result is thus attained. □

8. HADAMARD PRODUCT

Theorem 8.10. *If $f, g \in \mathcal{T}_m(\alpha, \mu, \lambda, \mathbb{T})$ then the Hadamard product $f * g \in \mathcal{T}_m(I, \mu, \lambda, \mathbb{T})$ where*

$$I = 1 - \frac{(1-\alpha)^2(n+1)(1-\mu)}{(1-\alpha)^2(1-\mu(n+1)) + (n+\alpha-\alpha\mu(n+1))^2)\Psi^m(\lambda, n)}. \tag{8.17}$$

Proof. By hypothesis of Theorem (2.1) we have

$$\sum_{n=1}^{\infty} \frac{[n+\alpha-\alpha\mu(n+1)]\Psi^m(\lambda, n)}{1-\alpha} a_n \leq 1 \tag{8.18}$$

and

$$\sum_{n=1}^{\infty} \frac{[n+\alpha-\alpha\mu(n+1)]\Psi^m(\lambda, n)}{1-\alpha} b_n \leq 1. \tag{8.19}$$

We find the largest I such that

$$\sum_{n=1}^{\infty} \frac{[n+I-I\mu(n+1)]\Psi^m(\lambda, n)}{1-I} a_n b_n \leq 1. \tag{8.20}$$

From (8.18) and (8.19), by using Cauchy-Schwarz inequality we find that

$$\sum_{n=1}^{\infty} \frac{[n+\alpha-\alpha\mu(n+1)]\Psi^m(\lambda, n)}{1-\alpha} \sqrt{a_n b_n} \leq 1. \tag{8.21}$$

We want to show that

$$\frac{[n+I-I\mu(n+1)]\Psi^m(\lambda, n)}{1-I} a_n b_n \leq \frac{[n+\alpha-\alpha\mu(n+1)]\Psi^m(\lambda, n)}{1-\alpha} \sqrt{a_n b_n}.$$

$$\Rightarrow \sqrt{a_n b_n} \leq \frac{(1-I)[n+\alpha-\alpha\mu(n+1)]}{(1-\alpha)[n+I-I\mu(n+1)]}. \tag{8.22}$$

Furthermore, from (8.21) we have

$$\sqrt{a_n b_n} \leq \frac{1-\alpha}{[n+\alpha-\alpha\mu(n+1)]\Psi^m(\lambda, n)}. \tag{8.23}$$

Thus from (8.22) and (8.23), we show that

$$\frac{1-\alpha}{[n+\alpha-\alpha\mu(n+1)]\Psi^m(\lambda, n)} \leq \frac{(1-I)[n+\alpha-\alpha\mu(n+1)]}{(1-\alpha)[n+I-I\mu(n+1)]}$$

which results in

$$I \leq \frac{[n+\alpha-\alpha\mu(n+1)]^2\Psi^m(\lambda, n) - n(1-\alpha)^2}{[n+\alpha-\alpha\mu(n+1)]^2\Psi^m(\lambda, n) + (1-\alpha)^2[1-\mu(n+1)]} = \varphi(n).$$

It is seen that $\varphi(n)$ is an increasing function of n $(n \geq 1)$. This proves the assertion on letting n=1. □

Theorem 8.11. *Let the functions $f, g \in \mathcal{T}_m(\alpha,\mu,\lambda,\mathbb{T})$. Then the function*

$q(z) = \dfrac{1}{z} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)z^n \in \mathcal{T}_m(\alpha,\mu,\lambda,\mathbb{T})$ *where*

$$I \leq 1 - \frac{2(1-\alpha)^2\Psi^m(\lambda,n)[1-\mu(n+1)+n]}{\{[n+\alpha-\alpha\mu(n+1)]\Psi^m(\lambda,n)\}^2 + 2(1-\alpha)^2\Psi^m(\lambda,n)[1-\mu(n+1)]}.$$

Proof. If $f, g \in \mathcal{T}_m(\alpha,\mu,\lambda,\mathbb{T})$, then

$$\sum_{n=1}^{\infty}\left[\frac{[n+\alpha-\alpha\mu(n+1)]\Psi^m(\lambda,n)}{1-\alpha}a_n\right]^2 \leq 1 \tag{8.24}$$

and

$$\sum_{n=1}^{\infty}\left[\frac{[n+\alpha-\alpha\mu(n+1)]\Psi^m(\lambda,n)}{1-\alpha}b_n\right]^2 \leq 1. \tag{8.25}$$

Combining the inequalities (8.24) and (8.25),

$$\sum_{n=1}^{\infty}\frac{1}{2}\left[\frac{[n+\alpha-\alpha\mu(n+1)]\Psi^m(\lambda,n)}{1-\alpha}\right]^2(a_n^2+b_n^2) \leq 1.$$

We find the largest I such that

$$\sum_{n=1}^{\infty}\left[\frac{[n+I-I\mu(n+1)]\Psi^m(\lambda,n)}{1-I}\right](a_n^2+b_n^2) \leq 1. \tag{8.26}$$

The inequality (8.26) holds if

$$\frac{[n+I-I\mu(n+1)]\Psi^m(\lambda,n)}{1-I} \leq \frac{1}{2}\left[\frac{[n+\alpha-\alpha\mu(n+1)]\Psi^m(\lambda,n)}{1-\alpha}\right]^2$$

$$\begin{aligned}\Rightarrow I &\leq \frac{([n+\alpha-\alpha\mu(n+1)]\Psi^m(\lambda,n))^2 - 2n(1-\alpha)^2\Psi^m(\lambda,n)}{([n+\alpha-\alpha\mu(n+1)]\Psi^m(\lambda,n))^2 - 2(1-\alpha)^2\Psi^m(\lambda,n)[1-\mu(n+1)]}\\ &= 1 - \frac{2(1-\alpha)^2\Psi^m(\lambda,n)[1-\mu(n+1)+n]}{([n+\alpha-\alpha\mu(n+1)]\Psi^m(\lambda,n))^2 - 2(1-\alpha)^2\Psi^m(\lambda,n)[1-\mu(n+1)]}.\end{aligned}$$

□

9. Conclusions

The introduced subclass $\mathcal{T}_m(\alpha,\mu,\lambda,\mathbb{T})$ on Hilbert Space operator theory had resulted in the study of necessary and sufficient condition for functions of the form (1.1) to belong to this class. Distortion theorem, radii results, Integral transforms and Hadamard Product property were examined. This also gives us the understanding that this study can further be extended with proper analysis on various other existing operators.

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