CREAT. MATH. INFORM.
Volume **31** (2022), No. 2,
Pages 173 - 184

Online version at `https://semnul.com/creative-mathematics/`
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2022.02.03

On a particular extension of the EV-Theorem

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ABSTRACT. The main aim of the paper is to determine the extreme values of the product $P = a_1a_2\cdots a_n$ under the constraints $\sum_{i=1}^n a_i = S$ and $\sum_{i=1}^n \frac{1}{a_i+1} = S_0$ for $n \ge 3$ nonnegative real numbers $a_1, a_2, \ldots, a_n$ and some given constants S and S_0. Some interesting applications of our results are provided as well.

1. INTRODUCTION

Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be nonnegative real numbers such that

$$\sum_{i=1}^n a_i = fixed, \qquad \sum_{i=1}^n \frac{1}{a_i+1} = fixed.$$

If we are interested in finding the minimum and the maximum value of the product

$$P = a_1a_2\cdots a_n,$$

then we are tempted to use the EV-Theorem (see [1-3]). To do this, the following substitution is necessary:

$$\frac{1}{a_i+1} = x_i, \qquad a_i = \frac{1}{x_i} - 1, \qquad x_i \in (0,1], \qquad i = 1, 2, \ldots, n.$$

Thus, we need to find the minimum and the maximum value of the product

$$P = \left(\frac{1}{x_1} - 1\right)\left(\frac{1}{x_2} - 1\right)\cdots\left(\frac{1}{x_n} - 1\right)$$

for

$$\sum_{i=1}^n x_i = fixed, \qquad \sum_{i=1}^n \frac{1}{x_i} = fixed.$$

By the EV-Theorem, if f is a real valued function, continue and differentiable on $(0,1)$, $f(1-) = \pm\infty$ and the joined function $g(x) = f'\left(\frac{1}{\sqrt{x}}\right)$ is strictly convex for $\frac{1}{\sqrt{x}} \in (0,1)$, i.e. for $x \in (1, \infty)$, then the sum

$$S_n = f(x_1) + f(x_2) + \cdots + f(x_n)$$

attains its maximum (if S_n has a global maximum) for $x_1 = x_2 = \cdots = x_{n-1} \le x_n$, and its minimum (if S_n has a global minimum) for $x_1 \le x_2 = x_3 = \cdots = x_n$. In our case, the function

$$f(x) = \ln\left(\frac{1}{x} - 1\right)$$

Received: 03.01.2022. In revised form: 28.04.2022. Accepted: 05.04.2022
2000 *Mathematics Subject Classification.* 26D07, 26D10, 26D15.
Key words and phrases. *EV-Theorem, maximum product, minimum product, fixed sums.*
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has

$$f'(x) = \frac{-1}{x(1-x)}, \quad g(x) = f'\left(\frac{1}{\sqrt{x}}\right) = \frac{-x}{\sqrt{x}-1},$$

$$g''(x) = \frac{\sqrt{x}-3}{4\sqrt{x}(\sqrt{x}-1)^3}.$$

Since g is not convex or concave on $(1, \infty)$, we cannot apply the EV-Theorem. Another similar example can be found in [4].

Note that the domain

$$D = \left\{(a_1, \ldots, a_n) \in \mathbb{R}_+^n : \sum_{i=1}^n a_i = S \geq 0, \sum_{i=1}^n \frac{1}{a_i+1} = S_0\right\}$$

is a non-empty compact set in $\mathbb{R}_+^n$ if and only if

$$\frac{n^2}{S+n} \leq S_0 \leq \frac{(n-1)S+n}{S+1}. \tag{*}$$

The left inequality turns out from the AM-HM inequality, while the right inequality can be obtained from Karamata's inequality [5,6] applied to the convex function $g(x) = \dfrac{1}{x+1}$, $x \geq 0$:

$$g(a_1) + g(a_2) + \cdots + g(a_n) \leq g(a_1 + a_2 + \cdots + a_n) + g(0) + \cdots + g(0).$$

Under the condition (*), there is a unique set $(a_1, a_2, \ldots, a_n)$ such that $a_1 \geq a_2 = a_3 = \cdots = a_n \geq 0$, $\sum_{i=1}^n a_i = S$ and $\sum_{i=1}^n \frac{1}{a_i+1} = S_0$. Also, under the condition

$$\frac{n^2}{S+n} \leq S_0 < \frac{S+n(n-1)}{S+n-1}, \quad S > 0, \tag{**}$$

there is a unique set $(a_1, a_2, \ldots, a_n)$ such that $a_1 = a_2 = \cdots = a_{n-1} \geq a_n > 0$, $\sum_{i=1}^n a_i = S$ and $\sum_{i=1}^n \frac{1}{a_i+1} = S_0$. Moreover, for

$$\frac{S+n(n-1)}{S+n-1} \leq S_0 \leq \frac{(n-1)S+n}{S+1}, \quad S \geq 0, \tag{***}$$

there is at least a set $(a_1, a_2, \ldots, a_n)$ such that $a_n = 0$, $\sum_{i=1}^n a_i = S$ and $\sum_{i=1}^n \frac{1}{a_i+1} = S_0$.

2. Main results

The main results of the paper are given in Theorem 2.1 and Theorem 2.2. To prove Theorem 2.1, we need Lemma 2.1 and Proposition 2.1 below.

Lemma 2.1. *Let a, b, c be nonnegative real numbers such that $a \geq b \geq c$ and*

$$a + b + c = S > 0, \quad \frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = S_0,$$

where $S_0 \in (1, 3)$ and $\dfrac{9}{S+3} < S_0 < \dfrac{2S+3}{S+1}$. For fixed S and S_0, the range of b is an interval $[m, M]$ with $m < M$. In addition, $b = m$ for $b = c$, and $b = M$ for either $a = b$ or $c = 0$.

Proof. From

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} < 1 + 1 + 1 = 3$$

we get $S_0 < 3$, by the AM-HM inequality

$$[(a+1) + (b+1) + (c+1)]\left[\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1}\right] \geq 9$$

we get $S_0 \geq \dfrac{9}{S+3}$, and from Karamata's inequality

$$\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1} \leq \frac{1}{a+b+c+1}+\frac{1}{0+1}+\frac{1}{0+1}$$

we get

$$S_0 \leq \frac{2S+3}{S+1}.$$

The equalities $S_0 = \dfrac{9}{S+3}$ and $S_0 = \dfrac{2S+3}{S+1}$ involve $a = b = c = \dfrac{S}{3}$ and $S = a > b = c = 0$, respectively. Therefore, in these cases, $m = M$. Next, according to the statement conditions, we may consider a and c ($a > c$) as functions of b. From

$$a' + 1 + c' = 0, \qquad \frac{a'}{(a+1)^2} + \frac{1}{(b+1)^2} + \frac{c'}{(c+1)^2} = 0,$$

we get

$$a'(b) = \frac{-(b-c)(b+c+2)(a+1)^2}{(a-c)(a+c+2)(b+1)^2} \leq 0, \qquad c'(b) = \frac{-(a-b)(a+b+2)(c+1)^2}{(a-c)(a+c+2)(b+1)^2} \leq 0.$$

Let us define the nonnegative functions

$$f_1(b) = b - c(b), \qquad f_2(b) = a(b) - b, \qquad f_3(b) = c(b).$$

Since

$$f_1'(b) = 1 - c'(b) > 0, \qquad f_2'(b) = a'(b) - 1 < 0, \qquad f_3'(b) = c'(b) \leq 0,$$

these functions are strictly increasing, decreasing and decreasing, respectively. The inequality $f_1(b) \geq 0$ (with f_1 increasing) involves $b \geq m$, where m is a root of the equation $c(b) = b$, the inequality $f_2(b) \geq 0$ (with f_2 decreasing) involves $b \leq b_2$, where b_2 is a root of the equation $a(b) = b$, and the inequality $f_3(b) \geq 0$ (with f_3 decreasing) involves $b \leq b_3$, where b_3 is a root of the equation $c(b) = 0$. Therefore, $M = \min\{b_2, b_3\}$ and $b \in [m, M]$, with $b = m$ for $b = c$, and $b = M$ for either $a = b$ or $c = 0$. $\square$

Proposition 2.1. *Let a_1, b_1, c_1 be fixed nonnegative real numbers,*

$$S = a_1 + b_1 + c_1, \qquad S_0 = \frac{1}{a_1+1} + \frac{1}{b_1+1} + \frac{1}{c_1+1},$$

and let a, b, c be nonnegative real numbers such that $a \geq b \geq c$ and

$$a + b + c = S, \qquad \frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = S_0.$$

For $S_0 > 1$, the product $P = abc$ achieves its maximum for $a \geq b = c$, and its minimum for either $a = b \geq c > 0$ or $c = 0$.

Proof. If $S = 0$, then $a = b = c = 0$ and the conclusion follows. Consider further $S > 0$. As shown at Lemma 2,1, in the special cases $S_0 = 3$, $S_0 = \dfrac{9}{S+3}$ and $S_0 = \dfrac{2S+3}{S+1}$, a single set (a, b, c) verifies the given equations. This set has respectively $a = b = c = 0$, $a = b = c = \dfrac{S}{3}$ and $S = a > b = c = 0$, satisfying the extremum conditions in the statement ($b = c$ and either $a = b$ or $c = 0$). Consider further that

$$S_0 < 3, \qquad S_0 > \frac{9}{S+3}, \qquad S_0 < \frac{2S+3}{S+1},$$

when $b \in [m, M]$, $m < M$. Thus, we may consider a and c as functions of b. We will show that $P'(b) \leq 0$. From

$$P'(b) = a'bc + ac + abc'$$

and the expressions of a' and c' determined in the proof of Lemma 2.1, we write the inequality $P'(b) \leq 0$ as

$$ab(a-b)(a+b+2)(c+1)^2 + bc(b-c)(b+c+2)(a+1)^2 \geq ac(a-c)(a+c+2)(b+1)^2.$$

Replacing $a-c$ with $(a-b)+(b-c)$, the inequality becomes as follows:

$$a(a-b)A \geq c(b-c)B,$$

where

$$A = b(S+2-c)(c+1)^2 - c(S+2-b)(b+1)^2,$$
$$B = a(S+2-b)(b+1)^2 - b(S+2-a)(a+1)^2.$$

Since

$$A = (S+2)[b(c+1)^2 - c(b+1)^2] + bc[(b+1)^2 - (c+1)^2]$$
$$= (S+2)(b-c)(1-bc) + bc(b-c)(S+2-a) = (b-c)(S+2-abc)$$

and

$$B = (S+2)[a(b+1)^2 - b(a+1)^2] + ab[(a+1)^2 - (b+1)^2]$$
$$= (S+2)(a-b)(1-ab) + ab(a-b)(S+2-c) = (a-b)(S+2-abc),$$

we have

$$a(a-b)A - c(b-c)B = (a-b)(b-c)(a-c)(S+2-abc).$$

Thus, we only need to show that $S+2-abc \geq 0$. Indeed, from $S_0 > 1$, we get

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} > 1,$$

which is equivalent to $S+2-abc > 0$. Since $P'(b) \leq 0$, the function $P(b)$ is strictly decreasing, therefore the product P achieves its maximum for $b = m$, when $a \geq b = c$, and its minimum for $b = M$, when either $a = b \geq c$ or $c = 0$ (see Lemma 2.1). □

Theorem 2.1. *Let $c_1, c_2, \ldots, c_n$ ($n \geq 3$) be fixed nonnegative real numbers,*

$$S = \sum_{i=1}^{n} c_i, \qquad S_0 = \sum_{i=1}^{n} \frac{1}{c_i+1},$$

and let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1 \geq a_2 \geq \cdots \geq a_n$ and

$$\sum_{i=1}^{n} a_i = S, \qquad \sum_{i=1}^{n} \frac{1}{a_i+1} = S_0.$$

If $S_0 > 1$ for $n = 3$ and $S_0 \geq n-2$ for $n \geq 4$, then

(a) the product $P = a_1a_2\cdots a_n$ achieves its maximum for $a_1 \geq a_2 = a_3 = \cdots = a_n$;

(b) the product $P = a_1a_2\cdots a_n$ achieves its minimum for either $a_1 = a_2 = \cdots = a_{n-1} \geq a_n > 0$ or $a_n = 0$.

Proof. Since the domain

$$D = \left\{(a_1, \ldots, a_n) \in \mathbb{R}_+^n : \sum_{i=1}^{n} a_i = S, \sum_{i=1}^{n} \frac{1}{a_i+1} = S_0\right\}$$

is a non-empty compact set in $\mathbb{R}_+^n$, the product P achieves its maximum and minimum. For $n = 3$, the conclusion follows from Proposition 2.1. For $n \geq 4$, we use the contradiction method.

(a) Assume, for the sake of contradiction, that P achieves its maximum at $(b_1, b_2, \ldots, b_n)$ with $b_1 \geq b_2 \geq \cdots \geq b_n$ and $b_2 > b_n$. Let x_1, x_2, x_n be nonnegative real numbers such that $x_1 \geq x_2 \geq x_n$ and

$$x_1 + x_2 + x_n = b_1 + b_2 + b_n,$$

$$\frac{1}{1+x_1}+\frac{1}{1+x_2}+\frac{1}{1+x_n}=\frac{1}{1+b_1}+\frac{1}{1+b_2}+\frac{1}{1+b_n}:=S_3.$$

We have

$$S_0=\sum_{i=1}^{n}\frac{1}{b_i+1}\le\frac{1}{1+b_1}+\frac{1}{1+b_2}+\frac{n-2}{1+b_n}\le S_3+n-3,$$

hence

$$S_3\ge S_0-n+3\ge n-2-n+3=1.$$

The equality $S_3=1$ holds only if $S_0=n-2$ and $b_3=b_4=\cdots=b_n=0$. This is not possible since it leads to the contradiction

$$n-2=\sum_{i=1}^{n}\frac{1}{b_i+1}=\frac{1}{1+b_1}+\frac{1}{1+b_2}+n-2.$$

Therefore, we have $S_3>1$. According to Proposition 2.1, the product $x_1x_2x_n$ achieves its maximum for $x_2=x_n$. So, we have $x_1x_2x_n>b_1b_2b_n$, which contradicts the assumption that the product achieves its maximum at $(b_1,b_2,\dots,b_n)$.

(b) Assume, for the sake of contradiction, that P achieves its minimum at $(b_1,b_2,\dots,b_n)$ with $b_1\ge b_2\ge\cdots\ge b_n>0$ and $b_1>b_{n-1}$. Let x_1,x_{n-1},x_n be nonnegative real numbers such that $x_1\ge x_{n-1}\ge x_n$ and

$$x_1+x_{n-1}+x_n=b_1+b_{n-1}+b_n,$$

$$\frac{1}{1+x_1}+\frac{1}{1+x_{n-1}}+\frac{1}{1+x_n}=\frac{1}{1+b_1}+\frac{1}{1+b_{n-1}}+\frac{1}{1+b_n}:=S_3.$$

We have

$$S_0=\sum_{i=1}^{n}\frac{1}{b_i+1}\le\frac{1}{1+b_1}+\frac{n-2}{1+b_{n-1}}+\frac{1}{1+b_n}\le S_3+n-3,$$

hence

$$S_3\ge S_0-n+3\ge n-2-n+3=1.$$

The equality $S_3=1$ holds only if $S_0=n-2$ and $b_2=b_3=\cdots=b_n=0$. This is not possible since it leads to the contradiction

$$n-2=\sum_{i=1}^{n}\frac{1}{b_i+1}=\frac{1}{1+b_1}+n-1.$$

Therefore, we have $S_3>1$. According to Proposition 2.1, the product $x_1x_{n-1}x_n$ achieves its minimum for $x_1=x_{n-1}>x_n>0$ or $x_n=0$. Thus, we have $x_1x_{n-1}x_n>b_1b_{n-1}b_n$, which contradicts the assumption that the product achieves its minimum at $(b_1,b_2,\dots,b_n)$. □

Lemma 2.2. *Let a,b,c be nonnegative real numbers such that $a\ge b\ge c$ and*

$$a+b+c=S,\qquad \frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}=S_0,$$

where $S_0<1$ and $S_0>\dfrac{9}{S+3}$. For fixed S and S_0, the range of b is an interval $[m,M]$ with $m<M$. In addition, $b=m$ for $b=c$, and $b=M$ for $a=b$.

Proof. It is not possible to have $c=0$ since this involves the contradiction

$$1>S_0=\frac{1}{a+1}+\frac{1}{b+1}+1.$$

By the AM-HM inequality

$$[(a+1)+(b+1)+(c+1)]\left[\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}\right]\geq 9,$$

we get

$$S_0 \geq \frac{9}{S+3}.$$

The equality $S_0 = \frac{9}{S+3}$ involves $a = b = c = \frac{S}{3}$, hence $m = M$. For $S_0 > \frac{9}{S+3}$, we may consider a and c as functions of b. Furthermore, the proof is identical to that of Lemma 2.1, but without using the function $f_3(b)$ (because it cannot decrease to zero). □

Proposition 2.2. *Let a_1, b_1, c_1 be fixed nonnegative real numbers,*

$$S = a_1 + b_1 + c_1, \qquad S_0 = \frac{1}{a_1+1}+\frac{1}{b_1+1}+\frac{1}{c_1+1},$$

and let a, b, c be nonnegative real numbers such that $a \geq b \geq c$ and

$$a+b+c = S, \qquad \frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1} = S_0.$$

For $S_0 < 1$, the product $P = abc$ achieves its maximum for $a = b \geq c$, and its minimum for $a \geq b = c$.

Proof. If $S = 0$, then $a = b = c = 0$ and the conclusion follows. Consider further $S > 0$. As shown at Lemma 2.2, in the special case $S_0 = \frac{9}{S+3}$, the given equations are satisfied for $a = b = c = \frac{S}{3}$. Consider further that $S_0 > \frac{9}{S+3}$, when $a > c$ and $b \in [m, M]$, $m < M$. Thus, we may consider a and c as functions of b. We will show that $P'(b) \geq 0$. As shown in the proof of Proposition 2.1, this inequality is equivalent to

$$(a-b)(b-c)(a-c)(S+2-abc) \leq 0.$$

Thus, we only need to show that $S + 2 - abc \leq 0$. Indeed, from $S_0 < 1$ we get

$$\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1} < 1,$$

which is equivalent to $S + 2 - abc < 0$. Since $P'(b) \geq 0$, the function $P(b)$ is strictly increasing, therefore the product P achieves its maximum for $b = M$, when $a = b \geq c$, and its minimum for $b = m$, when $a \geq b = c$ (see Lemma 2.2). □

Theorem 2.2. *Let $c_1, c_2, \ldots, c_n$ ($n \geq 3$) be fixed nonnegative real numbers,*

$$S = \sum_{i=1}^{n} c_i, \qquad S_0 = \sum_{i=1}^{n} \frac{1}{c_i+1},$$

and let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1 \geq a_2 \geq \cdots \geq a_n$ and

$$\sum_{i=1}^{n} a_i = S, \qquad \sum_{i=1}^{n} \frac{1}{a_i+1} = S_0.$$

If $S_0 < 1$ for $n = 3$ and $S_0 \leq 1$ for $n \geq 4$, then

(a) the product $P = a_1a_2\cdots a_n$ achieves its maximum for $a_1 = a_2 = \cdots = a_{n-1} \geq a_n$;

(b) the product $P = a_1a_2\cdots a_n$ achieves its minimum for $a_1 \geq a_2 = a_3 = \cdots = a_n$.

Proof. Since the domain

$$D = \left\{ (a_1, \ldots, a_n) \in \mathbb{R}_+^n : \sum_{i=1}^n a_i = S, \sum_{i=1}^n \frac{1}{a_i + 1} = S_0 \right\}$$

is a non-empty compact set in $\mathbb{R}_+^n$, the product P achieves its maximum and minimum. For $n = 3$, the conclusion turns out from Proposition 2.2. For $n \geq 4$, we use the contradiction method.

(a) Assume, for the sake of contradiction, that P has the maximum value for a set $(b_1, b_2, \ldots, b_n)$ with $b_1 \geq b_2 \geq \cdots \geq b_n$ and $b_1 > b_{n-1}$, which satisfies the given two equations. Let x_1, x_{n-1}, x_n be positive real numbers such that $x_1 \geq x_{n-1} \geq x_n$ and

$$x_1 + x_{n-1} + x_n = b_1 + b_{n-1} + b_n,$$

$$\frac{1}{1+x_1} + \frac{1}{1+x_{n-1}} + \frac{1}{1+x_n} = \frac{1}{1+b_1} + \frac{1}{1+b_{n-1}} + \frac{1}{1+b_n} := S_3.$$

We have

$$S_3 < S_0 \leq 1.$$

According to Proposition 2.2, the product $x_1 x_{n-1} x_n$ achieves its maximum for $x_1 = x_{n-1}$. In this case we have $x_1 x_{n-1} x_n > b_1 b_{n-1} b_n$, which contradicts the assumption that the product achieves its minimum at $(b_1, b_2, \ldots, b_n)$.

(b) Similarly, we can prove that P achieves its minimum for $a_1 \geq a_2 = a_3 = \cdots = a_n$. □

Remark 2.1. The problem of determining the maximum and minimum value of the product $P = a_1 a_2 \cdots a_n$ remains an open one for $1 < S_0 < n - 2$ (see Theorem 2.1) or $1 < m < n - 2$ (see Theorem 2.1′).

Remark 2.2. We may reformulate Theorem 2.1 and Theorem 2.2 as follows:

Theorem 2.1′. *Let $c_1, c_2, \ldots, c_n$ $(n \geq 3)$ be fixed nonnegative real numbers such that*

$$\sum_{i=1}^n \frac{1}{(n-m)c_i + m} = 1 ,$$

where $1 < m \leq 3$ for $n = 3$ and $n - 2 \leq m \leq n$ for $n \geq 4$. If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that $a_1 \geq a_2 \geq \cdots \geq a_n$ and

$$\sum_{i=1}^n a_i = \sum_{i=1}^n c_i, \quad \sum_{i=1}^n \frac{1}{(n-m)a_i + m} = 1,$$

then

(a) *the product $P = a_1 a_2 \cdots a_n$ achieves its maximum for $a_1 \geq a_2 = a_3 = \cdots = a_n$;*

(b) *the product $P = a_1 a_2 \cdots a_n$ achieves its minimum for either $a_1 = a_2 = \cdots = a_{n-1} \geq a_n > 0$ or $a_n = 0$.*

Theorem 2.2′. *Let $c_1, c_2, \ldots, c_n$ $(n \geq 3)$ be fixed nonnegative real numbers such that*

$$\sum_{i=1}^n \frac{1}{(n-m)c_i + m} = 1 ,$$

where $0 < m < 1$ for $n = 3$ and $0 < m \leq 1$ for $n \geq 4$. If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that $a_1 \geq a_2 \geq \cdots \geq a_n$ and

$$\sum_{i=1}^n a_i = \sum_{i=1}^n c_i, \quad \sum_{i=1}^n \frac{1}{(n-m)a_i + m} = 1,$$

then

(a) *the product* $P = a_1a_2\cdots a_n$ *achieves its maximum for* $a_1 = a_2 = \cdots = a_{n-1} \ge a_n$;

(b) *the product* $P = a_1a_2\cdots a_n$ *achieves its minimum for* $a_1 \ge a_2 = a_3 = \cdots = a_n$.

3. Applications

Application 3.1. *If* $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ *are nonnegative real numbers such that*

$$\sum_{i=1}^{n} \frac{1}{a_i + n - 1} = 1,$$

then

$$(n-2)(a_1 + a_2 + \cdots + a_n) + a_1a_2\cdots a_n \ge (n-1)^2.$$

Proof. Consider $a_1 \ge a_2 \ge \cdots \ge a_n$. According to Theorem 2.1′ (case $m = n - 1$), for fixed $a_1 + a_2 + \cdots + a_n$, the product $a_1a_2\cdots a_n$ has the minimum value for either $a_1 = a_2 = \cdots = a_{n-1} \ge a_n > 0$ or $a_n = 0$. Thus, it suffices to consider these cases.

Case 1: $a_1 = a_2 = \cdots = a_{n-1} \ge a_n > 0$. We need to show that if

$$\frac{n-1}{x+1} + \frac{1}{y+1} = 1\ ,$$

which leads to

$$y = \frac{n-1-(n-2)x}{x}\ , \qquad 0 < y \le x < \frac{n-1}{n-2}\ ,$$

then

$$(n-2)[(n-1)x + y] + x^{n-1}y \ge (n-1)^2,$$

which is equivalent to

$$(n-2)y + x^{n-1}y \ge (n-1)[n-1-(n-2)x].$$

Since $n - 1 - (n-2)x = xy$, we only need to show that

$$n - 2 + x^{n-1} \ge (n-1)x,$$

which is just the AM-GM inequality.

Case 2: $a_n = 0$. We need to show that

$$\sum_{i=1}^{n-1} \frac{1}{a_i + n - 1} = \frac{n-2}{n-1}$$

involves

$$(n-2)(a_1 + a_2 + \cdots + a_{n-1}) \ge (n-1)^2.$$

This follows immediately from the AM-HM inequality

$$\left[\sum_{i=1}^{n-1}(a_i + n - 1)\right]\left(\sum_{i=1}^{n-1}\frac{1}{a_i + n - 1}\right) \ge (n-1)^2.$$

The proof is completed. The equality occurs for $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = a_2 = \cdots = a_{n-1} = \dfrac{n-1}{n-2}$ and $a_n = 0$ (or any cyclic permutation).

Application 3.2. *If* $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ *are nonnegative real numbers such that*

$$\sum_{i=1}^{n} \frac{1}{2a_i + n - 2} = 1,$$

then

$$a_1 + a_2 + \cdots + a_n - n \ \geq\ 2^{n-1}(a_1 a_2 \cdots a_n - 1).$$

Proof. Consider $a_1 \geq a_2 \geq \cdots \geq a_n$. For $n = 3$, the inequality is an identity. For $n \geq 4$, according to Theorem 2.1′ (case $m = n - 2$), for fixed $a_1 + a_2 + \cdots + a_n$, the product $a_1 a_2 \cdots a_n$ attains its maximum value when $a_1 \geq a_2 = a_3 = \cdots = a_n$. Thus, we only need to show that

$$y + (n-1)x - n \geq 2^{n-1}(yx^{n-1} - 1)$$

for

$$\frac{1}{2y + n - 2} + \frac{n-1}{2x + n - 2} = 1\ ,$$

which implies

$$y = \frac{n - 2 - (n-3)x}{2x - 1}\ , \qquad \frac{1}{2} < x \leq y.$$

The required inequality is equivalent to

$$\frac{n - 2 - (n-3)x + (2x-1)[(n-1)x - n]}{2^{n-1}} \geq$$

$$\geq (n-2)(x^{n-1} - 1) - (n-3)(x^n - 1) - 2(x-1), \qquad (*)$$

or

$$\frac{(n-1)(x-1)^2}{2^{n-2}} \geq (x-1)f(x),$$

where

$$f(x) = (n-2)(x^{n-2} + x^{n-3} + \cdots + x + 1) - (n-3)(x^{n-1} + x^{n-2} + \cdots + x + 1) - 2$$

$$= (n-2)[(x^{n-2}-1)+(x^{n-3}-1)+\cdots+(x-1)] - (n-3)[(x^{n-1}-1)+(x^{n-2}-1)+\cdots+(x-1)]$$

$$= (x-1)g(x),$$

$$g(x) = (n-2)[x^{n-3} + 2x^{n-4} + \cdots + (n-2)] - (n-3)[x^{n-2} + 2x^{n-3} + \cdots + (n-1)]$$

$$= -(n-3)x^{n-2} - (n-4)x^{n-3} - \cdots - x^2 + 1.$$

So, we only need to show that

$$\frac{n-1}{2^{n-2}} \geq g(x).$$

Since g is a decreasing function, it suffices to show that

$$\frac{n-1}{2^{n-2}} \geq g\left(\frac{1}{2}\right).$$

This is true if the inequality (*) holds for $x = \dfrac{1}{2}$. It is easy to show that this last inequality is an identity.

For $n \geq 4$, the equality occurs when $a_1 = a_2 = \cdots = a_n = 1$.

Application 3.3. *If* $a_1, a_2, \ldots, a_n$ *($n \geq 3$) are nonnegative real numbers such that*

$$\sum_{i=1}^{n} \frac{1}{(n-1)a_i + 1} = 1,$$

then

$$a_1 + a_2 + \cdots + a_n - n \leq k(a_1 a_2 \cdots a_n - 1), \qquad k = \left(\frac{n-1}{n-2}\right)^{n-1}.$$

Proof. Consider $a_1 \geq a_2 \geq \cdots \geq a_n$. For $n = 3$, the inequality is an identity. Consider further $n \geq 4$. According to Theorem 2.2′ (case $m = 1$), for fixed $a_1 + a_2 + \cdots + a_n$, the product $a_1a_2\cdots a_n$ attains its minimum when $a_1 \ge a_2 = a_3 = \cdots = a_n$. We need to show that if

$$\frac{1}{(n-1)y+1} + \frac{n-1}{(n-1)x+1} = 1\,,$$

which leads to

$$y = \frac{1}{(n-1)x-n+2}\,, \qquad \frac{n-2}{n-1} < x \le y,$$

then

$$y + (n-1)x - n \le k(yx^{n-1} - 1),$$

which is equivalent to

$$1 + [(n-1)x-n+2][(n-1)x-n] \le k[x^{n-1} - (n-1)x + n - 2], \tag{**}$$

or

$$(n-1)^2(x-1)^2 \le kf(x), \qquad f(x) = x^{n-1} - 1 - (n-1)(x-1).$$

Since

$$f(x) = (x-1)(x^{n-2} + x^{n-3} + \cdots + x - n + 2) = (x-1)^2 g(x),$$

where

$$g(x) = x^{n-3} + 2x^{n-4} + \cdots + (n-2),$$

we only need to show that

$$(n-1)^2 \le kg(x).$$

Since g is an increasing function, it suffices to show that

$$(n-1)^2 \le kg\left(\frac{n-2}{n-1}\right).$$

This inequality is true if the inequality (**) holds for $x = \dfrac{n-2}{n-1}$. Indeed, in this case, (**) is an identity.

For $n \ge 4$, the equality occurs when $a_1 = a_2 = \cdots = a_n = 1$.

Remark 3.3. By the AM-HM inequality

$$\left[\sum_{i=1}^{n-1}((n-1)a_i + 1)\right]\left(\sum_{i=1}^{n-1}\frac{1}{(n-1)a_i+1}\right) \ge n^2,$$

we get $a_1 + a_2 + \cdots + a_n \ge n$. As a consequence, the inequality in Application 3.3 involves

$$a_1a_2\cdots a_n \ge 1.$$

Actually, the following stronger inequality holds for $n \ge 4$:

$$a_1a_2\cdots a_n \ge \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Indeed, denoting $p = a_1a_2\cdots a_n$ ($p \ge 1$), the inequality in Application 3.3 leads to

$$na_1a_2\cdots a_n - (a_1 + a_2 + \cdots + a_n) \ge np - k(p-1) - n = (n-k)(p-1) \ge 0.$$

Application 3.4. *If $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ are nonnegative real numbers such that*

$$\sum_{i=1}^{n}\frac{1}{(n-1)a_i+1} = 1.$$

then

$$a_1 + a_2 + \cdots + a_n \ge n\sqrt[n-1]{a_1a_2\cdots a_n}.$$

Proof. Consider $a_1 \geq a_2 \geq \cdots \geq a_n$. If $n \geq 4$, we may apply Theorem 2.2′ for $m = 1$. So, for fixed $a_1 + a_2 + \cdots + a_n$, the product $a_1 a_2 \cdots a_n$ has the maximum value when $a_1 \geq a_2 = a_3 = \cdots = a_n$, and we only need to show that if

$$\frac{n-1}{(n-1)x+1} + \frac{1}{(n-1)y+1} = 1, \quad x \leq y,$$

that is

$$y = \frac{1}{(n-1)x - n + 2}, \quad x > \frac{n-2}{n-1},$$

then

$$(n-1)x + y \geq n \sqrt[n-1]{x^{n-1}y}\ ,$$

that is

$$\left[\frac{(n-1)x+y}{nx}\right]^{n-1} \geq y\ , \quad \left(1 - \frac{x-y}{nx}\right)^{n-1} \geq y\ .$$

By Bernoulli's inequality, it suffices to show that

$$1 - \frac{(n-1)(x-y)}{nx} \geq y\ ,$$

which is equivalent to

$$x \geq (nx - n + 1)y,$$
$$(n-1)(x-1)^2 \geq 0.$$

For $n = 3$, we need to show that

$$a_1 + a_2 + a_3 \geq 3\sqrt{a_1 a_2 a_3}$$

for

$$\frac{1}{2a_1+1} + \frac{1}{2a_2+1} + \frac{1}{2a_3+1} = 1,$$

that is

$$4a_1 a_2 a_3 = a_1 + a_2 + a_3 + 1.$$

Denote $t = \sqrt[3]{a_1 a_2 a_3}$. From AM-GM inequality, we have

$$4t^3 = a_1 + a_2 + a_3 + 1 \geq 3t + 1,$$

hence $t \geq 1$. Finally, we get

$$a_1 + a_2 + a_3 - 3\sqrt{a_1 a_2 a_3} = 4t^3 - 1 - 3t\sqrt{t} = (t\sqrt{t} - 1)(4t\sqrt{t} + 1) \geq 0.$$

The equality occurs for $a_1 = a_2 = \cdots = a_n = 1$.

Application 3.5. *If a, b, c, d are nonnegative real numbers such that*

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} + \frac{1}{d+1} = 2$$

then

$$(a+b+c+d)^2 + 4 \geq 5(abc + bcd + cda + dab).$$

Proof. Consider $a \geq b \geq c \geq d$ and write the hypothesis in the form

$$a + b + c + d + 2 = abcd + 2(abc + bcd + cda + dab).$$

If the sum $a+b+c+d$ is fixed, then the expression $abc+bcd+cda+dab$ has the maximum value when the product $abcd$ has the minimum value, that is when either $a = b = c \geq d > 0$ or $d = 0$ (Theorem 2.1). Thus, it suffices to consider these cases.

Case 1: $a = b = c \geq d > 0$. We need to prove that

$$(3a+d)^2 + 4 \geq 5a^2(a+3d)$$

for

$$\frac{3}{a+1} + \frac{1}{d+1} = 2,$$

that is

$$d = \frac{2-a}{2a-1}, \qquad \frac{1}{2} < a \le 2.$$

Write the required inequality as follows:

$$(3a+d)^2 - 16 \ge 5(a^3 + 3a^2d - 4),$$

$$\frac{12(a-1)^2(a+1)(3a-1)}{(2a-1)^2} \ge \frac{10(a-1)^2(a^2+2)}{2a-1}.$$

It is true if

$$6(a+1)(3a-1) \ge 5(2a-1)(a^2+2).$$

Indeed, we have

$$6(a+1)(3a-1) - 5(2a-1)(a^2+2) \ge 6(a+1)(3a-1) - 5(2a-1)(2a+2) = 2(a+1)(2-a) \ge 0.$$

Case 2: $d = 0$. Let $s = a + b + c$. We need to show that

$$s^2 + 4 \ge 5abc$$

for

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = 1,$$

that is $abc = s + 2$. From

$$s^3 \ge 27abc = 27(s+2),$$

we get $(s-6)(s+3)^2 \ge 0$, hence $s \ge 6$. Finally,

$$s^2 + 4 - 5abc = s^2 + 4 - 5(s+2) = (s-6)(s+1) \ge 0.$$

The equality occurs for $a = b = c = d = 1$, and also for $a = b = c = 2$ and $d = 0$ (or any cyclic permutation).

Acknowledgement. The authors thank to the referee for his pertinent comments and suggestions which help us to improve the manuscript.

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