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Distance based topological descriptors for two classes of graphs

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ABSTRACT. In this paper, the exact formula for the generalized product degree distance, reciprocal product degree distance and product degree distance of Mycielskian graph and its complement are obtained. In addition, we compute the above indices for non-commuting graph.

1. INTRODUCTION

For vertices $u, v \in V(G)$, the distance between u and v in G, denoted by $d_G(u,v)$, is the length of a shortest (u,v)-path in G and let $d_G(v)$ be the degree of a vertex $v \in V(G)$. The *diameter* of the graph G is $max\{d_G(u,v)|u, v \in V(G)\}$. The *neighbor* of the vertex $u \in V(G)$ is $N_G(u) = \{v|uv \in E(G)\}$. A *topological index* of a graph is a real number related to the graph; it does not depend on labeling or pictorial representation of a graph. In theoretical chemistry, molecular structure descriptors (also called topological indices) are used for modeling physicochemical, pharmacologic, toxicologic, biological and other properties of chemical compounds [12]. There exist several types of such indices, especially those based on vertex and edge distances. One of the most intensively studied topological indices is the Wiener index.

Let G be a connected graph. Then $Wiener\ index$ of G is defined as $W(G) = \frac{1}{2} \sum\limits_{u,\, v \,\in\, V(G)} d_G(u,v)$ with the summation going over all pairs of distinct vertices of G. This definition can be further generalized in the following way: $W_\lambda(G) = \frac{1}{2} \sum\limits_{u,\, v \,\in\, V(G)} d_G^\lambda(u,v)$, where $d_G^\lambda(u,v) = (d_G(u,v))^\lambda$ and λ is a real number [13, 14]. If $\lambda = -1$, then $W_{-1}(G) = H(G)$, where $H(G)$ is Harary index of G. In the chemical literature also $W_{\frac{1}{2}}$ [35] as well as the general case W_λ were examined [9, 15].

Dobrynin and Kochetova [5] and Gutman [11] independently proposed a vertex-degree-weighted version of Wiener index called *degree distance*, which is defined for a connected graph G as $DD(G) = \frac{1}{2} \sum\limits_{u,v \in V(G)} (d_G(u) + d_G(v)) d_G(u,v)$. The *additively weighted Harary index*(H_A) or *reciprocal degree distance*(RDD) is defined in [1] as $H_A(G) = RDD(G) = \frac{1}{2} \sum\limits_{u,v \in V(G)} \frac{(d_G(u)+d_G(v))}{d_G(u,v)}$. Hua and Zhang [18] have obtained lower and upper bounds for the reciprocal degree distance of graph in terms of other graph invariants. The chemical applications and mathematical properties of the reciprocal degree distance are well studied in [1, 22, 31].

The *generalized degree distance*, denoted by $H_\lambda(G)$, is defined as $H_\lambda(G) = \frac{1}{2} \sum\limits_{u,v \in V(G)} (d_G(u) + d_G(v)) d_G^\lambda(u,v)$, where λ is a any real number. If $\lambda = 1$, then $H_\lambda(G) = DD(G)$ and if

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$\lambda = -1$, then $H_\lambda(G) = RDD(G)$. The generalized degree distance of unicyclic and bicyclic graphs are studied by Hamzeh et. al [16, 17]. Also they are given the generalized degree distance of Cartesian product, join, symmetric difference, composition and disjunction of two graphs. The generalized degree distance of the strong and tensor product of graphs are obtained in [27, 28]. In this sequence, the *generalized product degree distance*, denoted by $H^*_\lambda(G)$, is defined as $H^*_\lambda(G) = \frac{1}{2}\sum\limits_{u,v\in V(G)} d_G(u)d_G(v)d^\lambda_G(u,v)$. If $\lambda = 1$, then $H^*_\lambda(G) = DD_*(G)$ and if $\lambda = -1$, then $H_\lambda(G) = RDD_*(G)$. Therefore the study of the above topological indices are important and we try to obtain the results related to this index. In this paper, the exact formulae for the generalized product degree distance, reciprocal product degree distance and product degree distance of Mycielskian graph and its complement are obtained. In addition, we compute the above indices for non-commuting graph.

The *first Zagreb index* is defined as $M_1(G) = \sum\limits_{u\in V(G)} d_G(u)^2$ and the *second Zagreb index* is defined as $M_2(G) = \sum\limits_{uv\in E(G)} d_G(u)d_G(v)$. In fact, one can rewrite the first Zagreb index as $M_1(G) = \sum\limits_{uv\in E(G)} (d_G(u) + d_G(v))$. Similarly, the *first Zagreb coindex* is defined as $\overline{M}_1(G) = \sum\limits_{uv\notin E(G)} (d_G(u) + d_G(v))$ and the *second Zagreb coindex* is defined as $\overline{M}_2(G) = \sum\limits_{uv\in E(G)} d_G(u)d_G(v)$. The Zagreb indices are found to have applications in QSPR and QSAR studies as well, see [7].

2. Distance based topological index

In this section, we obtain the exact formulae for some distance based topological indices, such as generalized product degree distance, product degree distance and reciprocal product degree distance of Mycielskian graph and its complement. The *maximum* and *minimum* degree of the graph G are denoted by Δ and δ, respectively.

2.1. Bounds for RDD_*.

For a complete graph K_n, we have $RDD_*(K_n) > RDD(K_n)$ and for a star graph S_n, $RDD_*(S_n) < RDD(S_n)$. Now we obtain the sharp lower and upper bounds for $RDD_*(G)$.

Theorem 2.1. *Let G be a connected graph on n vertices. Then $RDD(G)-H(G) \leq RDD_*(G) \leq RDD(G) + \Delta(\Delta-2)H(G)$, with equality holds for both lower and upper bounds if and only if G is isomorphic to a star graph S_n and G is a regular graph, respectively.*

Proof. One can observe that

$$\begin{aligned} RDD_*(G) - RDD(G) &= \sum_{u,v\in V(G)} \Big(\frac{d_G(u)d_G(v) - d_G(u) - d_G(v)}{d_G(u,v)}\Big) \\ &= \sum_{u,v\in V(G)} \Big(\frac{(d_G(u)-1)(d_G(v)-1)-1}{d_G(u,v)}\Big) \end{aligned}$$

For each vertex $x \in V(G)$, we have $\delta(x) \leq \Delta$. Hence

$$RDD_*(G) - RDD(G) \leq \sum_{u,v\in V(G)} \frac{(\Delta-1)^2-1}{d_G(u,v)} = \Delta(\Delta-2)H(G).$$

Thus $RDD_*(G) \leq RDD(G) + \Delta(\Delta-2)H(G)$.

Similarly, by the definitions of $RDD_*(G)$ and $RDD(G)$, we have

$$\begin{aligned}
RDD_*(G)-RDD(G) &= \sum_{u,v\in V(G)}\Big(\frac{(d_G(u)-1)(d_G(v)-1)-1}{d_G(u,v)}\Big)\\
&= \sum_{u,v\in V(G),\ d_G(u)=1}\Big(\frac{(d_G(u)-1)(d_G(v)-1)-1}{d_G(u,v)}\Big)\\
&\quad + \sum_{u,v\in V(G),\ d_G(u)\geq 2,\ d_G(v)\geq 2}\Big(\frac{(d_G(u)-1)(d_G(v)-1)-1}{d_G(u,v)}\Big)\\
&\geq \sum_{u,v\in V(G),\ d_G(u)=1}\Big(\frac{(d_G(u)-1)(d_G(v)-1)-1}{d_G(u,v)}\Big)\\
&= -\sum_{u,v\in V(G),\ d_G(u)=1}\frac{1}{d_G(u,v)} \qquad (2.1)
\end{aligned}$$

$$\begin{aligned}
H(G) &= \sum_{u,v\in V(G),\ d_G(u)=1}\frac{1}{d_G(u,v)}+\sum_{u,v\in V(G),\ d_G(u)\geq 2,\ d_G(v)\geq 2}\frac{1}{d_G(u,v)}\\
&\leq \sum_{u,v\in V(G),\ d_G(u)=1}\frac{1}{d_G(u,v)}. \qquad (2.2)
\end{aligned}$$

From (2.1) and (2.2), we have $RDD_*(G)\geq RDD(G)-H(G)$.

The equality holds for lower bound (resp. upper bound) if and only if $G\cong S_n$(resp. G is regular).

Using above theorem, we have the following corollary. □

Corollary 2.1. *Let G be connected graph on n vertices. Then $RDD_*(G)\leq RDD(G)+(n-1)(n-3)H(G)$, with equality if and only if $G\cong K_n$.*

2.2. **Mycielskian graph.** In a search for triangle-free graphs with arbitrarily large chromatic number, Mycielski [25] developed an interesting graph transformation as follows. Let G be a connected graph with vertex set $V(G)=\{v_1,v_2,\ldots,v_n\}$. The *Mycielskian graph*$\mu(G)$ of G contains G itself as an isomorphic subgraph, together with $n+1$ additional vertices: a vertex u_i corresponding to each vertex v_i of G, and another vertex w. Each vertex u_i is connected by an edge to w, so that these vertices form a subgraph in the form of a star $K_{1,n}$. The Mycielskian and generalized Mycielskians have fascinated graph theorists a great deal. This has resulted in studying several graph parameters of these graphs, see [10]. In recent times, there has been an increasing interest in the study of Mycielskian graph [6, 4, 21]. The generalized degree distance of the Mycielskian is obtained in [29]. In this section, generalized product degree distance of Mycielskian graph is obtained.

The following lemmas are follows from the structure of the Mycielskian of the given graph.

Remark 2.1. Let G be a graph with $V(G)=\{v_1,v_2,\ldots,v_n\}$. Then there are $n-1$ two element subsets in $V(G)$. Therefore

$$\sum_{\{v_i,v_j\}\subseteq V(G)}\Big(d_G(v_i)+d_G(v_j)\Big)=\sum_{i=1}^{n}(n-1)d_G(v_i)=2(n-1)m.$$

Remark 2.2. Let G be a graph n vertices and m edges. Then

$$\begin{aligned}(2m)^2 = \Big(\sum_{i=1}^{n} d_G(v_i)\Big)^2 &= \sum_{i=1}^{n} d_G^2(v_i) + 2\sum_{\{v_i,v_j\}\subseteq V(G)} d_G(v_i)d_G(v_j)\\ &= M_1(G) + 2\sum_{\{v_i,v_j\}\subseteq V(G)} d_G(v_i)d_G(v_j).\end{aligned}$$

Thus $\sum\limits_{\{v_i,v_j\}\subseteq V(G)} d_G(v_i)d_G(v_j) = 2m^2 - \frac{M_1(G)}{2}$.

Lemma 2.1. Let G be a connected graph. Then the distances between the vertices of the Mycielskian graph $\mu(G)$ of G are given as follows. For each $x, y \in V(\mu(G))$,

$$(i)\ d^{\lambda}_{\mu(G)}(x,y) = \begin{cases} 2^{\lambda}\ if\ x = u_i, y = u_j \\ d^{\lambda}_G(v_i,v_j)\ if\ x = v_i, y = v_j, d_G(v_i,v_j) \le 3 \\ 4^{\lambda}\ if\ x = v_i, y = v_j, d_G(v_i,v_j) \ge 4. \end{cases}$$

$$(ii)\ d^{\lambda}_{\mu(G)}(x,y) = \begin{cases} 2^{\lambda}\ if\ x = v_i, y = u_j, i = j \\ d^{\lambda}_G(v_i,v_j)\ if\ u = v_i, v = x_j, i \neq j, d_G(v_i,v_j) \le 2 \\ 3^{\lambda}\ if\ x = v_i, y = u_j, i \neq j, d_G(v_i,v_j) \ge 3. \end{cases}$$

$$(iii)\ d^{\lambda}_{\mu(G)}(x,y) = \begin{cases} 1\ if\ x = u_i, y = w \\ 2^{\lambda}\ if\ x = v_i, y = w. \end{cases}$$

Lemma 2.2. *Let G be a graph with n vertices. Then the degree of the vertex $x \in \mu(G)$ is*

$$d_{\mu(G)}(x) = \begin{cases} n\ if\ x = w \\ 1 + d_G(v_i)\ if\ x = u_i \\ 2d_G(v_i)\ if\ x = v_i. \end{cases}$$

Theorem 2.2. *Let G be a graph on n vertices and m edges with diameter 2. Then $H^*_{\lambda}(\mu(G)) = 8H^*_{\lambda}(G) + 2H_{\lambda}(G) + (n^2 + 2mn) + 2^{\lambda}\Big(\frac{3M_1(G)}{2} + \frac{n(n-1)}{2} + 2m(m + 3n + 1)\Big)$.*

Proof. From the structure of the Mycielskian graph, we consider the following cases of adjacent and nonadjacent pairs of vertices in $\mu(G)$ to compute $H_{\lambda}(\mu(G))$.

• If $\{x, y\} \subseteq U$, then

$$\begin{aligned}&\sum_{\{u_i,u_j\}\subseteq U} d_{\mu(G)}(u_i)d_{\mu(G)}(u_j)d^{\lambda}_{\mu(G)}(u_i,u_j)\\ =&\sum_{\{u_i,u_j\}\subseteq U} (1 + d_G(v_i))(1 + d_G(v_j))2^{\lambda},\\ &\text{by Lemmas 2.1 and 2.2}\\ =&\ 2^{\lambda}\sum_{\{i,j\}\subseteq[n]} \Big(1 + (d_G(v_i) + d_G(v_j)) + d_G(v_i)d_G(v_j)\Big)\end{aligned}$$

By Remarks 2.1 and 2.2, we have

$$\sum_{\{u_i,u_j\}\subseteq U} d_{\mu(G)}(u_i)d_{\mu(G)}(u_j)d^{\lambda}_{\mu(G)}(u_i,u_j) = 2^{\lambda}\Big(\frac{n(n-1)}{2} + 2m(n-1) + 2m^2 - \frac{M_1(G)}{2}\Big).$$

• If $\{x, y\} \subseteq V(G)$, then $d_{\mu(G)}(v_i, v_j) = d_G(v_i, v_j)$ for each $v_i, v_j \in V(G)$. Hence

$$\begin{aligned}\sum_{\{v_i,v_j\}\subseteq V(G)} d_{\mu(G)}(v_i)d_{\mu(G)}(v_j)d^{\lambda}_{\mu(G)}(v_i,v_j) &= \sum_{\{v_i,v_j\}\subseteq V(G)} 4d_G(v_i)d_G(v_j)d^{\lambda}_G(v_i,v_j), \\ &\quad \text{by Lemmas 2.1 and 2.2} \\ &= 4H^*_{\lambda}(G).\end{aligned}$$

• If $x = v_i$ and $y = u_i,\ 1 \le i \le n$, then

$$\begin{aligned}\sum_{i=1}^{n} d_{\mu(G)}(v_i)d_{\mu(G)}(u_i)d^{\lambda}_{\mu(G)}(v_i,u_i) &= \sum_{i=1}^{n} 2d_G(v_i)(1+d_G(v_i))2^{\lambda},\ \text{by Lemmas 2.1 and 2.2} \\ &= 2^{\lambda}\Big(4m + 2M_1(G)\Big).\end{aligned}$$

• If $x = v_i$ and $y = u_j,\ i \neq j$, then

$$\begin{aligned}&\sum_{\{v_i,u_j\}\subseteq V(\mu(G)),\, i\neq j} d_{\mu(G)}(v_i)d_{\mu(G)}(u_j)d^{\lambda}_{\mu(G)}(v_i,u_j) \\ =& \sum_{\{v_i,u_j\}\subseteq V(\mu(G)),\, i\neq j} 2d_G(v_i)(1+d_G(v_j))d^{\lambda}_{\mu(G)}(v_i,u_j),\ \text{by Lemma 2.2} \\ =&\ 2\sum_{\{v_i,u_j\}\subseteq V(\mu(G)),\, i\neq j} d_G(v_i)d^{\lambda}_{\mu(G)}(v_i,u_j) \\ &+2\sum_{\{v_i,u_j\}\subseteq V(\mu(G)),\, i\neq j} d_G(v_i)d_G(v_j)d^{\lambda}_{\mu(G)}(v_i,u_j) \\ =&\ S_1 + S_2,\ \text{where } S_1 \text{ and } S_2 \text{ are the sums in order.} \qquad (2.3)\end{aligned}$$

Now we obtain S_1 and S_2 are separately.

$$\begin{aligned}S_1 &= 2\sum_{\{v_i,u_j\}\subseteq V(\mu(G)),\, i\neq j} d_G(v_i)d^{\lambda}_{\mu(G)}(v_i,u_j), \\ &\quad \text{since } d^{\lambda}_{\mu(G)}(v_i,u_j) = d^{\lambda}_{\mu(G)}(v_j,u_i) \text{ and by Lemma 2.1} \\ &= 2\sum_{\{v_i,u_j\}\subseteq V(\mu(G))} d_G(v_i)d^{\lambda}_{\mu(G)}(v_i,v_j) \\ &= 2\sum_{\{i,j\}\subseteq[n]} (d_G(v_i)+d_G(v_j))d^{\lambda}_G(v_i,v_j) \\ &= 2H_{\lambda}(G). \qquad (2.4)\end{aligned}$$

$$\begin{aligned}S_2 &= 2\sum_{\{v_i,u_j\}\subseteq V(\mu(G))} d_G(v_i)d_G(v_j)d^{\lambda}_{\mu(G)}(v_i,u_j) \\ &= 4\sum_{\{i,j\}\subseteq[n]} d_G(v_i)d_G(v_j)d^{\lambda}_G(v_i,v_j) \\ &= 4H^*_{\lambda}(G). \qquad (2.5)\end{aligned}$$

Using (2.4) and (2.5) in (2.3), we have

$$\sum_{\{v_i,u_j\}\subseteq V(\mu(G)),\, i\neq j} d_{\mu(G)}(v_i)d_{\mu(G)}(u_j)d^{\lambda}_{\mu(G)}(v_i,u_j) = 2H_{\lambda}(G) + 4H^*_{\lambda}(G).$$

• If $x = w$ and $y \in U$, then

$$\begin{aligned}\sum_{i=1}^{n} d_{\mu(G)}(w)d_{\mu(G)}(u_i)d^{\lambda}_{\mu(G)}(w,u_i) &= \sum_{i=1}^{n} n(d_G(v_i)+1), \text{ by Lemmas 2.1 and 2.2}\\ &= n^2+2mn.\end{aligned}$$

• If $x = w$ and $y \in V(G)$, then

$$\begin{aligned}\sum_{i=1}^{n} d_{\mu(G)}(w)d_{\mu(G)}(v_i)d^{\lambda}_{\mu(G)}(x,v_i) &= \sum_{i=1}^{n} 2nd_G(v_i)\,2^{\lambda}, \text{ by Lemmas 2.1 and 2.2}\\ &= 2^{\lambda}(4mn).\end{aligned}$$

□

Summarizing the total contributions of above cases of adjacent and nonadjacent pairs of vertices in $\mu(G)$, we can obtain the desired result. This completes the proof.

Using $\lambda = 1$ in Theorem 2.2, we obtain the product degree distance of the Mycielskian graph.

Corollary 2.2. *Let G be a graph on n vertices and m edges with diameter 2. Then $DD_*(\mu(G)) = 8DD_*(G) + 2DD(G) + 3M_1(G) + n(2n-1) + 2m(m+7n+2)$.*

Using $\lambda = -1$ in Theorem 2.2, we obtain the reciprocal product degree distance of the Mycielskian graph.

Corollary 2.3. *Let G be a graph on n vertices and m edges with diameter 2. Then $RDD_*(\mu(G)) = 8RDD_*(G) + 2RDD(G) + \frac{3M_1(G)}{4} + \frac{n(5n-4)}{4} + m(m+5n+1)$.*

2.3. **Complement of the Mycielskian graph.** The following lemmas are follows from the structure of the complement of the Mycielskian graph.

Lemma 2.3. *Let G be a connected graph. Then the distances between the vertices of the Mycielskian graph $\overline{\mu}(G)$ of G are given as follows. For each $x, y \in V(\overline{\mu}(G))$,*

$$(i)\ d^{\lambda}_{\overline{\mu}(G)}(x,y) = \begin{cases} 1 \ if \ x = u_i, y = u_j \\ 1 \ if \ x = v_i, y = v_j, d_G(v_i,v_j) > 1 \\ 2^{\lambda} \ if \ x = v_i, y = v_j, d_G(v_i,v_j) = 1. \end{cases}$$

$$(ii)\ d^{\lambda}_{\overline{\mu}(G)}(x,y) = \begin{cases} 1 \ if \ x = v_i, y = u_j, i = j \\ 1 \ if \ x = v_i, y = u_j, i \neq j, d_G(v_i,v_j) > 1 \\ 2^{\lambda} \ if \ x = v_i, y = u_j, i \neq j, d_G(v_i,v_j) = 1. \end{cases}$$

$$(iii)\ d^{\lambda}_{\overline{\mu}(G)}(x,y) = \begin{cases} 2^{\lambda} \ if \ x = u_i, y = w \\ 1 \ if \ x = v_i, y = w. \end{cases}$$

Lemma 2.4. *Let G be a graph on n vertices. Then the degree of the vertex $x \in \overline{\mu}(G)$ is*

$$d_{\overline{\mu}(G)}(x) = \begin{cases} n \ if \ x = w \\ 2n - 1 - d_G(v_i) \ if \ x = u_i \\ 2n - 2d_G(v_i) \ if \ x = v_i. \end{cases}$$

Theorem 2.3. *Let G be a graph on n vertices and m edges with diameter 2. Then* $H^*_\lambda(\overline{\mu}(G)) = 2^\lambda\Big(4M_2(G) - 2nM_1(G)(2n+3) + 2n^3 + (12m-1)n^2 + 8m^2 - 6mn\Big) - \frac{5M_1(G)}{2} + \Big(8n^4 - 6n^3 + \frac{5n^2}{2} + 18m^2 + 14nm - 24n^2m - 2m - \frac{n}{2}\Big)$.

Proof. From the structure of the complement of Mycielskian graph, we consider the following cases of adjacent and nonadjacent pairs of vertices in $\overline{\mu}(G)$ to compute $H_\lambda(\overline{\mu}(G))$.
• If $\{x, y\} \subseteq U$, then

$$\begin{aligned} & \sum_{\{u_i,u_j\}\subseteq U} d_{\overline{\mu}(G)}(u_i) d_{\overline{\mu}(G)}(u_j) d^\lambda_{\overline{\mu}(G)}(u_i,u_j) \\ = & \sum_{\{u_i,u_j\}\subseteq U} \Big(2n-1-d_G(v_i))(2n-1-d_G(v_j)), \\ & \text{by Lemmas 2.3 and 2.4} \\ = & \sum_{\{i,j\}\subseteq[n]} \Big((2n-1)^2 - (2n-1)(d_G(v_i)+d_G(v_j)) + d_G(v_i)d_G(v_j)\Big) \end{aligned}$$

By Remarks 2.1 and 2.2, we have

$$\begin{aligned} = & \frac{n(n-1)(2n-1)^2}{2} - 2m(2n-1)(n-1) + 2m^2 - \frac{M_1(G)}{2} \\ = & \frac{(n-1)(2n-1)}{2}(2n^2-n-4m) + 2m^2 - \frac{M_1(G)}{2}. \end{aligned}$$

• If $\{x, y\} \subseteq V(G)$, then $d^\lambda_{\overline{\mu}(G)}(v_i, v_j) = 1$ for each $v_iv_j \notin E(G)$ and $d^\lambda_{\overline{\mu}(G)}(v_i, v_j) = 2^\lambda$ otherwise. Moreover $\{\{v_i, v_j\} \subseteq V(G) : i \neq j, v_iv_j \notin E(G)\} = \{\{v_i, v_j\} \subseteq V(G) : i \neq j\} \setminus \{\{v_i, v_j\} \subseteq V(G) : v_iv_j \in E(G)\}$.

$$\begin{aligned} & \sum_{\{v_i,v_j\}\subseteq V(G)} d_{\overline{\mu}(G)}(v_i) d_{\overline{\mu}(G)}(v_j) d^\lambda_{\overline{\mu}(G)}(v_i,v_j) \\ = & \sum_{v_iv_j\notin E(G)} (2n-2d_G(v_i))(2n-2d_G(v_j)) \\ & + \sum_{v_iv_j\in E(G)} (2n-2d_G(v_i))(2n-2d_G(v_j))2^\lambda, \\ & \text{by Lemmas 2.3 and 2.4} \\ = & \sum_{\{v_i,v_j\}\subseteq V(G)} \Big(4n^2 - 4n(d_G(v_i)+d_G(v_j)) + 4d_G(v_i)d_G(v_j)\Big) \\ & + \sum_{v_iv_j\in E(G)} \Big(4n^2 - 4n(d_G(v_i)+d_G(v_j)) + 4d_G(v_i)d_G(v_j)\Big)2^\lambda \end{aligned}$$

By Remarks 2.1 and 2.2, we have

$$\begin{aligned} = & 4n^2\Big(\frac{n(n-1)}{2}\Big) - 8mn(n-1) + 4(2m^2 - \frac{M_1(G)}{2}) \\ & +2^\lambda\Big(4mn^2 - 4nM_1(G) + 4M_2(G)\Big) \\ = & 2n(n-1)(n^2-4m) + 8m^2 - 2M_1(G) \\ & +2^\lambda\Big(4mn^2 - 4nM_1(G) + 4M_2(G)\Big) \end{aligned}$$

• If $x = v_i$ and $y = u_i,\ 1 \leq i \leq n$, then

$$\begin{aligned}\sum_{i=1}^{n} d_{\overline{\mu}(G)}(v_i) d_{\overline{\mu}(G)}(u_i) d^{\lambda}_{\overline{\mu}(G)}(v_i, u_i) &= \sum_{i=1}^{n} (2n - 2d_G(v_i))(2n - 1 - d_G(v_i)), \\ &\quad \text{by Lemmas 2.3 and 2.4} \\ &= 2n^2(2n-1) - 2m(6n-2) + 2M_1(G).\end{aligned}$$

• If $x = v_i$ and $y = u_j,\ i \neq j$, then $\{(v_i, v_j) : i \neq j, v_iv_j \notin E(G)\} = \{(v_i, v_j) : i \neq j\} \setminus \{(v_i, v_j) : v_iv_j \in E(G)\}$. Thus

$$\begin{aligned}&\sum_{\{v_i,u_j\} \subseteq V(\overline{\mu}(G)),\, i \neq j} d_{\overline{\mu}(G)}(v_i) d_{\overline{\mu}(G)}(u_j) d^{\lambda}_{\overline{\mu}(G)}(v_i, u_j) \\ &= \sum_{(v_i,v_j),\, v_iv_j \notin E(G)} (2n - 2d_G(v_i))(2n - 1 - d_G(v_j)) \\ &\quad + \sum_{(v_i,v_j),\, v_iv_j \in E(G)} (2n - 2d_G(v_i))(2n - 1 - d_G(v_j))2^{\lambda}, \text{ by Lemmas 2.3 and 2.4} \\ &= \sum_{(v_i,v_j), i \neq j} (2n - 2d_G(v_i))(2n - 1 - d_G(v_j)) \\ &\quad + 2^{\lambda} \sum_{(v_i,v_j),\, v_iv_j \in E(G)} (2n - 2d_G(v_i))(2n - 1 - d_G(v_j))\end{aligned}$$

Each v_j can be paired with $n-1$ vertices v_i as $(v_i, v_j),\ i \neq j,\ \sum\limits_{(v_i,v_j)} d_G(v_j) = (n-1)\sum\limits_{j=1}^{n} d_G(v_j) = 2m(n-1)$. Moreover, $\sum\limits_{(v_i,v_j)} d_G(v_i)d_G(v_j) = 2\sum\limits_{\{v_i,v_j\}} d_G(v_i)d_G(v_j)$. Since $|\{(v_i, v_j) : i \neq j\}| = n(n-1)$, then by Remarks 2.1 and 2.2, we have

$$\begin{aligned}\sum_{(v_i,v_j), i \neq j} (2n - 2d_G(v_i))(2n - 1 - d_G(v_j)) &= 2n(2n-1)n(n-1) - 2n(n-1)2m \\ &\quad - 2(2n-1)(n-1)2m + 4\left(2m^2 - \frac{M_1(G)}{2}\right) \\ &= (2n^2 - 4m)(2n-1)(n-1) - 4n(n-1)m \\ &\quad + 8m^2 - 2M_1(G). \qquad (2.6)\end{aligned}$$

One can observe that $|\{(v_i, v_j) : v_iv_j \in E(G)\}| = 2m$ and $\sum\limits_{(v_i,v_j),\ v_iv_j \in E(G)} d_G(v_i) = \sum\limits_{i=1}^{n} (d_G(v_i))^2$, since each v_i has $d_G(v_i)$ neighbors and appears $d_G(v_i)$ times. Then by Remarks 2.1 and 2.2, we have

$$\begin{aligned}&2^{\lambda} \sum_{(v_i,v_j),\, v_iv_j \in E(G)} (2n - 2d_G(v_i))(2n - 1 - d_G(v_j)) \\ &= 2^{\lambda}\Big(2n(2n-1)2m - 2nM_1(G) - 2(2n-1)M_1(G) \\ &\quad + 4(2m^2 - \frac{M_1(G)}{2})\Big) \\ &= 2^{\lambda}(4n(2n-1)m + 8m^2 - 6nM_1(G)). \qquad (2.7)\end{aligned}$$

From (2.6) and (2.7) we have

$$\begin{aligned}&\sum_{\{v_i,u_j\}\subseteq V(\overline{\mu}(G)),\, i\neq j} d_{\overline{\mu}(G)}(v_i)d_{\overline{\mu}(G)}(u_j)d^{\lambda}_{\overline{\mu}(G)}(v_i,u_j)\\ =\;& (2n^2-4m)(2n-1)(n-1)-4n(n-1)m+8m^2\\ &-2M_1(G)+2^{\lambda}(4n(2n-1)m+8m^2-6nM_1(G)).\end{aligned}$$

• If $x = w$ and $y \in U$, then

$$\begin{aligned}\sum_{i=1}^{n} d_{\overline{\mu}(G)}(w)d_{\overline{\mu}(G)}(u_i)d^{\lambda}_{\overline{\mu}(G)}(w,u_i) &= \sum_{i=1}^{n} n(2n-1-d_G(v_i))2^{\lambda}, \text{ by Lemmas 2.3 and 2.4}\\ &= 2^{\lambda}\Big(n^2(2n-1)-2mn\Big).\end{aligned}$$

• If $x = w$ and $y \in V(G)$, then

$$\begin{aligned}\sum_{i=1}^{n} d_{\overline{\mu}(G)}(w)d_{\overline{\mu}(G)}(v_i)d^{\lambda}_{\overline{\mu}(G)}(w,v_i) &= \sum_{i=1}^{n} n(2n-2d_G(v_i)), \text{ by Lemmas 2.3 and 2.4}\\ &= 2n^3-4mn.\end{aligned}$$

□

Summarizing the total contributions of above cases of adjacent and nonadjacent pairs of vertices in $\mu(G)$, we can obtain the desired result. This completes the proof.

Using $\lambda = 1$ in Theorem 2.3, we obtain the product degree distance of the complement of the Mycielskian graph.

Corollary 2.4. *Let G be a graph on n vertices and m edges with diameter 2. Then* $DD_*(\overline{\mu}(G)) = 8M_2(G) - (16n^2+24n+5)\frac{M_1(G)}{2} + \Big(8n^4 - 2n^3 + \frac{n^2}{2} + 34m^2 + 2mn - 2m - \frac{n}{2}\Big)$.

Using $\lambda = -1$ in Theorem 2.3, we obtain the reciprocal product degree distance of the complement of the Mycielskian graph.

Corollary 2.5. *Let G be a graph on n vertices and m edges with diameter 2. Then* $RDD_*(\overline{\mu}(G)) = 2M_2(G) - (4n^2+6n+5)\frac{M_1(G)}{2} + \Big(8n^4 - 5n^3 + 2n^2 + 22m^2 - 11mn - 2m - 18n^2m - \frac{n}{2}\Big)$.

3. Non-commuting graph

Let G be a non-abelian group and let $Z(G)$ be the center of G. The *non-commuting graph* $\Gamma(G)$ is a graph obtained from the group G with $V(\Gamma(G)) = G \setminus Z(G)$ and $E(\Gamma(G)) = \{uv | uv \neq vu\}$. This concept was introduced by Neumann [26] in 1976. The graph $\Gamma(G)$ has exactly $|G|-|Z(G)|$ vertices and $\frac{|G|}{2}(|G|-k(G))$ edges, where $k(G)$ denotes the number of conjugacy classes of G. The complement of a graph Γ is a graph $\overline{\Gamma}$ on the same vertices such that two vertices of $\overline{\Gamma}$ are adjacent if and only if they are not adjacent in Γ. The complement graph $\overline{\Gamma}(G)$ is called the commuting graph of G. For more details, see [2, 3, 23, 24].

Theorem 3.4. *Let G be a non-abelian finite group. Then* $H_{\lambda}(\Gamma(G)) = 2^{\lambda+1}(|G| - |Z(G)| - 1)\,|E(\Gamma(G))| - (2^{\lambda}-1)M_1(\Gamma(G))$.

Proof. Let $\Gamma(G)$ be a non-commutating graph of G with exactly n vertices. By the definition of H_λ,

$$\begin{aligned} H_\lambda(\Gamma(G)) &= \sum_{\{u,v\}\subseteq V(\Gamma(G))} (d_{\Gamma(G)}(u)+d_{\Gamma(G)}(v))d^{\lambda}_{\Gamma(G)}(u,v) \\ &= \sum_{uv\in E(\Gamma(G))} (d_{\Gamma(G)}(u)+d_{\Gamma(G)}(v)) + 2^{\lambda} \sum_{uv\in E(\overline{\Gamma}(G))} (d_{\Gamma(G)}(u)+d_{\Gamma(G)}(v)). \end{aligned}$$

For any vertex $u \in \overline{\Gamma}(G)$, the degree of u is $d_{\Gamma(G)}(u) = |G| - |Z(G)| - 1 - d_{\overline{\Gamma}(G)}(u)$. From the definition of first Zagreb index, we have

$$\begin{aligned} H_\lambda(\Gamma(G)) &= M_1(\Gamma(G)) + 2^{\lambda} \sum_{uv\in E(\overline{\Gamma}(G))} \Big(2(|G|-|Z(G)|-1) - (d_{\overline{\Gamma}(G)}(u)+d_{\overline{\Gamma}(G)}(v))\Big) \\ &= M_1(\Gamma(G)) + 2^{\lambda+1}\left|E(\overline{\Gamma}(G))\right|(|G|-|Z(G)|-1) - 2^{\lambda} \sum_{u\in V(\Gamma(G))} (d_{\overline{\Gamma}(G)}(u))^2 \\ &= M_1(\Gamma(G)) + 2^{\lambda+1}\left|E(\overline{\Gamma}(G))\right|(|G|-|Z(G)|-1) \\ &\quad -2^{\lambda} \sum_{u\in V(\Gamma(G))} (|G|-|Z(G)|-1-d_{\Gamma(G)}(u))^2 \\ &= M_1(\Gamma(G)) + 2^{\lambda+1}\left|E(\overline{\Gamma}(G))\right|(|G|-|Z(G)|-1) \\ &\quad -2^{\lambda} \sum_{u\in V(\Gamma(G))} \Big((|G|-|Z(G)|-1)^2 + (d_{\Gamma(G)}(u))^2 - 2(|G|-|Z(G)|-1)d_{\Gamma(G)}(u)\Big) \\ &= M_1(\Gamma(G)) + 2^{\lambda+1}\left|E(\overline{\Gamma}(G))\right|(|G|-|Z(G)|-1) \\ &\quad -2^{\lambda}\Big((|V(G)|-|Z(G)|-1)^2((|G|-|Z(G)|)+M_1(\Gamma(G))-4(|G|-|Z(G)|-1)\left|E(\overline{\Gamma}(G))\right|\Big) \\ &= (1-2^{\lambda})M_1(\Gamma(G)) + 2^{\lambda+1}\left|E(\overline{\Gamma}(G))\right|(|G|-|Z(G)|-1) \\ &\quad -2^{\lambda}(|G|-|Z(G)|-1)^2(|G|-|Z(G)|) + 2^{\lambda+2}(|G|-|Z(G)|-1)\left|E(\Gamma(G))\right|. \end{aligned}$$

Since $\left|E(\overline{\Gamma}(G))\right| = \frac{(|G|-|Z(G)|)(|G|-|Z(G)|-1)}{2} - |E(\Gamma(G))|$, we obtain

$$H_\lambda(\Gamma(G)) = 2^{\lambda+1}(|G|-|Z(G)|-1)\left|E(\Gamma(G))\right| - (2^{\lambda}-1)M_1(\Gamma(G)).$$

By using $\lambda = 1$ in Theorem 3.4, we obtain the degree distance of the graph $\Gamma(G)$. □

Corollary 3.6. *Let G be a non-abelian finite group. Then $DD(\Gamma(G)) = 4\left|E(\Gamma(G))\right|(|G| - |Z(G)| - 1) - M_1(\Gamma(G))$.*

By using $\lambda = -1$ in Theorem 3.4, we obtain the reciprocal degree distance of the graph $\Gamma(G)$.

Corollary 3.7. *Let G be a non-abelian finite group. Then $RDD(\Gamma(G)) = \left|E(\Gamma(G))\right|(|G| - |Z(G)| - 1) + \frac{M_1(\Gamma(G))}{2}$.*

Theorem 3.5. *Let G be a non-abelian finite group. Then $H^*_\lambda(\Gamma(G)) = 2^{\lambda+1}\left|E(\Gamma(G))\right|^2 - 2^{\lambda-1}M_1(\Gamma(G)) - (2^{\lambda}-1)M_2(\Gamma(G))$.*

Proof. It follows from that $\overline{M}_2(\Gamma(G)) = \sum_{uv \in E(\overline{\Gamma}(G))} d_{\Gamma(G)}(u) d_{\Gamma(G)}(v)$ and $\overline{M}_2(\Gamma(G)) = 2\,|E(\Gamma(G))|^2 - M_2(\Gamma(G)) - \frac{M_1(\Gamma(G))}{2}$. From the definition of H_λ^*,

$$\begin{aligned}
H_\lambda^*(\Gamma(G)) &= \sum_{\{u,v\} \subseteq V(\Gamma(G))} d_{\Gamma(G)}(u) d_{\Gamma(G)}(v) d^\lambda_{\Gamma(G)}(u,v) \\
&= \sum_{uv \in E(\Gamma(G))} d_{\Gamma(G)}(u) d_{\Gamma(G)}(v) + 2^\lambda \sum_{uv \in E(\overline{\Gamma}(G))} d_{\Gamma(G)}(u) d_{\Gamma(G)}(v) \\
&= M_2(\Gamma(G)) + 2^\lambda \overline{M}_2(\Gamma(G)) \\
&= M_2(\Gamma(G)) + 2^\lambda \Big(2\,\big|E(\overline{\Gamma}(G))\big|^2 - M_2(\Gamma(G)) - \frac{M_1(\Gamma(G))}{2} \Big) \\
&= 2^{\lambda+1}\,|E(\Gamma(G))|^2 - 2^{\lambda-1} M_1(\Gamma(G)) - (2^\lambda - 1) M_2(\Gamma(G)).
\end{aligned}$$

By using $\lambda = 1$ in Theorem 3.4, we obtain the product degree distance of the graph $\Gamma(G)$. □

Corollary 3.8. *Let G be a non-abelian finite group. Then $DD^*(\Gamma(G)) = 4\,|E(\Gamma(G))|^2 - M_1(\Gamma(G)) - M_2(\Gamma(G))$.*

By using $\lambda = -1$ in Theorem 3.4, we obtain the reciprocal product degree distance of the graph $\Gamma(G)$.

Corollary 3.9. *Let G be a non-abelian finite group. Then $RDD^*(\Gamma(G)) = |E(\Gamma(G))|^2 - \frac{M_1(\Gamma(G))}{4} + \frac{M_2(\Gamma(G))}{2}$.*

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