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About the cube polynomial of Extended Fibonacci Cubes

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ABSTRACT. The hypercube is one of the best model for the network topology of a distributed system. In this paper we determine the cube polynomial of Extended Fibonacci Cubes, which is the counting polynomial for the number of induced k-dimensional hypercubes in Extended Fibonacci Cubes.

1. INTRODUCTION

An important aspect of designing a distributed system regards the design of the communication subsystem, that means the design of its interconnection network. The design of the interconnection network suppose a compromise to achieve some objectives as: high transfer rate, small communication delay, simplicity, scalability, optimal rapport cost/performance. An interconnection network consists of a set of processors, each with a local memory and a set of bidirectional (or unidirectional) links that serve for the exchange of data between processors. This network can be modeled as a finite graph $G = (V, E)$, with V the set of vertices and E the set of edges. The vertices of the graph represent the nodes of the network, that is processing elements, and the edges correspond to the communication links. If the communication between processors is unidirectional then the graph is a directed graph, otherwise the graph is undirected. Two processors connected by a link in the network are called neighbours. The interconnection graph of the network is referred as the network topology. Some of the key features of interest in such an interconnection network are its topological properties such as node degree, diameter, connectivity, structure, the embeddability of other topologies and the communication algorithms.

A number of interconnection topologies have been proposed in the literature. A widely studied interconnection topology is the hypercube, or the n-cube, H_n. The hypercube has good properties such as symmetry, small diameter and node degree, recursive structure, efficient communication algorithms. A drawback in the case of the hypercube is the number of nodes, which is a power of 2, and limits the choice for a network interconnection with a given number of nodes.

In [2] Hsu proposed and studied the properties of an interconnection topology called Fibonacci cube, based on the Fibonacci numbers. The Fibonacci cube can emulate many of the basic algorithms for the hypercube and there are more Fibonacci numbers in a given interval than powers of 2. In [6] Wu generalized the Fibonacci cube topology by defining the series of Extended Fibonacci Cubes, $(EFC_k)_{k>0}$. The Extended Fibonacci Cubes are also defined using the same recursive relation as the Fibonacci numbers, but changing the initial conditions. In this way the number of choices for the number of nodes for an interconnection network increases. One of the measures of "goodness" of a network topology is the embeddability, the capability to simulate other topologies in the network. In [1] Brešar et al. introduced the cube polynomial for a graph, the coefficients of this polynomial representing the number of induced hypercubes in the graph. In [4] Klavžar

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and Mollard determined the cube polynomials for Fibonacci and Lucas cubes. In this paper, we determine the cube polynomial for Extended Fibonacci Cubes.

2. Preliminaries

The Fibonacci Cube topology is based on the properties of Fibonacci numbers. The Fibonacci numbers are defined as $f_0 = 0, f_1 = 1, f_n = f_{n-1} + f_{n-2}$, for $n > 1$. According to Zeckendorf's lemma, any integer number $k, 0 \leq k \leq f_n$ can be written as a sum of Fibonacci numbers,

$$k = \sum_{i=2}^{n-1} b_i f_i,\ b_i \in \{0,1\}, b_i b_{i+1} = 0, i = \overline{1, n-2}.$$

This means that to every integer number $k, 0 \leq k \leq f_n$ we can associate a Fibonacci code $k_F = (b_{n-1} \dots b_3 b_2)_F$ according to its representation as a sum of Fibonacci numbers. This code is a binary code which has no two consecutive 1's and is called Fibonacci code. Any binary string with no two consecutive 1's is called Fibonacci string. The Fibonacci cube can be defined as follows:

Definition 2.1. The Fibonacci cube of order $n > 1$, denoted by Γ_n, is defined as the graph $\Gamma_n = (V_n, E_n)$ where the set of nodes is $V_n = \{0, 1, \dots, f_n - 1\}$ and the set of edges E_n is $E_n = \{(i,j) | H(i_F, j_F) = 1, i, j \in V_n\}$, where $H(i_F, j_F)$ is the Hamming distance between the Fibonacci codes of nodes i and j.

The Fibonacci cube of order n has f_n nodes and there is an edge between two nodes if their Fibonacci codes differ exactly in one position. A recursive definition for the Fibonacci cubes has been given in [2] as follows:

Definition 2.2. The Fibonacci cube $\Gamma_n = (V_n, E_n)$ of order $n, n > 1$, is defined recursively as $V_n = 0||V_{n-1} \cup 10||V_{n-2}$, where V_{n-1} and V_{n-2} are the set of nodes of the order $n-1$ respectively $n-2$ Fibonacci cubes and $||$ denotes the concatenation of strings and there is an edge between two nodes if their binary representations differ exactly in one position. The initial conditions are $\Gamma_2 = (\{\lambda\}, \varnothing)$ and $\Gamma_3 = (\{0,1\}, \{(0,1)\})$.

A Fibonacci cube of order n has f_n nodes and can be recursively decomposed in two Fibonacci cubes of order $n-1$ and $n-2$. The two Γ_{n-1} and Γ_{n-2} are connected by f_{n-2} edges. The Fibonacci cube has good properties: the nodes degree is between $n/8$ and $n-2$, the diameter is $n-2$, the node and edge connectivity are between $n/8$ and $(n-2)/3$ respectively, basic topologies such as arrays, rings, meshes, hypercubes can be embedded in Fibonacci cubes. But just the Fibonacci cubes with an even number of nodes, greater than 2 are hamiltonian. Wu introduced in [6] the series of Extended Fibonacci Cubes $(EFC_k)_{k>1}$, using a recursive definition and the properties of this interconnection topology were studied in [3] and [5]. Extended Fibonacci Cubes can be defined using extended Fibonacci numbers, obtained from Fibonacci numbers by changing the initial values.

Definition 2.3. The series of Extended Fibonacci Cubes, $(EFC_k)_{k>1}$, is defined as the graphs $EFC_k(n) = (V_k(n), E_k(n)), n > k+1$, where $V_k(n) = 0||V_k(n-1) \cup 10||V_k(n-2)$, $n > k+3$ and two nodes are connected by an edge in $E_k(n)$ if their binary representations differ in exactly one position. The initial conditions for $n = k+2$, respectively $n = k+3$ are $V_k(k+2) = \{0,1\}^k, V_k(k+3) = \{0,1\}^{k+1}$, where $\{0,1\}^k$ denotes the set of binary strings of length k.

From this definition we can see that an Extended Fibonacci Cube of type $k, EFC_k(n)$ can be decomposed in two extended Fibonacci cubes $EFC_k(n-1)$ and $EFC_k(n-2)$ and each node in $EFC_k(n-2)$ is connected to a node in $EFC_k(n-1)$.

The nodes of $EFC_k(n)$ are labelled with binary strings of length $n-2$, where the first $n-k-2$ bits represent a Fibonacci code and the last k represent any binary code. The number of nodes in $EFC_k(n)$ is $2^k f_{n-k}$, where f_{n-k} is the $(n-k)$-th Fibonacci number, $n > k+1$. The Extended Fibonacci Cubes $EFC_1(3), EFC_1(4), EFC_1(5)$ and $EFC_2(6)$ are given in Fig. 1.

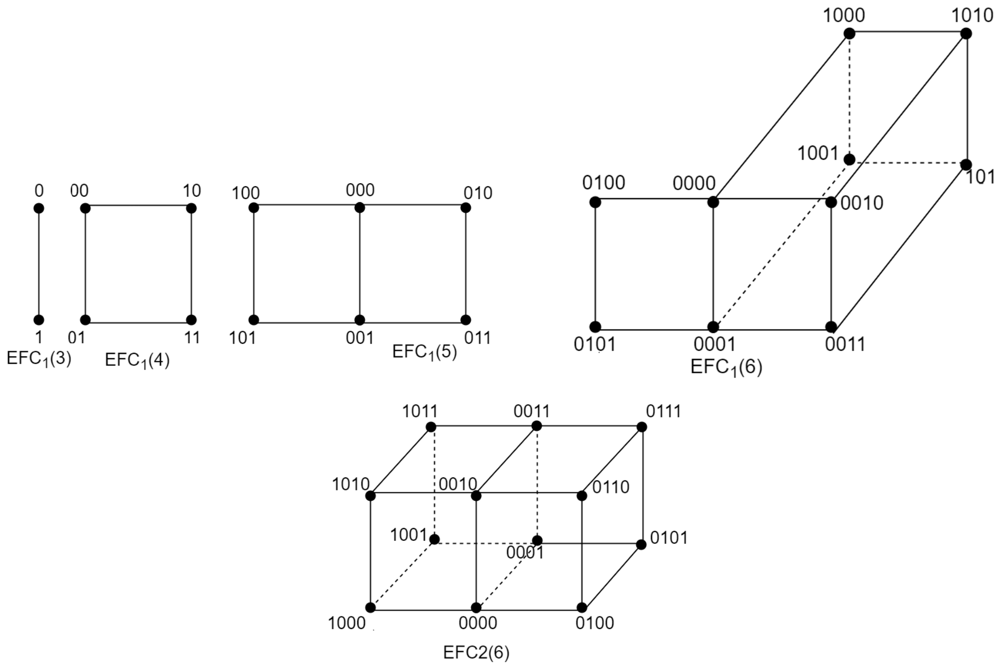

Fig. 1. Extended Fibonacci Cubes $EFC_1(3), EFC_1(4), EFC_1(5), EFC_1(6), EFC_2(6)$

The Extended Fibonacci Cubes have an even number of nodes and are all hamiltonian, the diameter of $EFC_k(n)$ is $n-2$, the degree $g(u)$ of a node u in $EFC_k(n)$ satisfies $\lceil \frac{n-(k-1)}{3} \rceil + (k-1) \le g(u) \le n-2$, $EFC_{n-2}(n) = H_{n-2}$ for $n > 2$ and we have that $\Gamma_n = EFC_0(n) \subset EFC_1(n) \subset \ldots \subset EFC_{n-2}(n) = H_{n-2}$.

In [6] Wu gave some embeddings of hypercubes and trees in $EFC_k(n) : EFC_k(n)$ is a subgraph of H_{n-2} respectively $H_k = EFC_k(k+2), H_{k+1} = EFC_K(k+3)$ and H_n is a proper subgraph of $EFC_k(2n-k+1)$ for $n > k+2$, that means a hypercube H_n can be embedded in any $EFC_k(2n-k+1)$ with dilation and congestion 1, $n > k+2$. Arrays and meshes can be also embedded in Extended Fibonacci Cubes, see [7].

Definition 2.4. For $k \ge 1$, the extended Fibonacci numbers of type k, are defined as the numbers $ef_n^{(k)}, n \ge k+2$, with $ef_n^{(k)} = ef_{n-1}^{(k)} + ef_{n-2}^{(k)}, n \ge k+4$, $ef_{k+2}^{(k)} = 2^k, ef_{k+3}^{(k)} = 2^{k+1}$.

For $k = 1$, we denote the extended Fibonacci numbers $ef_n, n \ge 3$ and we call them extended Fibonacci numbers. We have that $ef_n = ef_{n-1} + ef_{n-2}, n \ge 5\,, ef_3 = 2, ef_4 = 4$.

Let $\{ef_n\}_{n\ge3}$ be the extended Fibonacci numbers.

Proposition 2.1. *The generating function of the sequence $\{ef_n\}_{n\ge3}$ is*

$$s(x) = \sum_{n\ge3} ef_n \cdot x^n = \frac{2(x^4+x^3)}{1-x-x^2}.$$

Proof.

$$s(x) = \sum_{n\geq 3} ef_n \cdot x^n = ef_3 \cdot x^3 + ef_4 \cdot x^4 + \sum_{n\geq 5} ef_n \cdot x^n =$$
$$= 2 \cdot x^3 + 4 \cdot x^4 + \sum_{n\geq 5}(ef_{n-1} + ef_{n-2}) \cdot x^n =$$
$$= 2 \cdot x^3 + 4 \cdot x^4 + \sum_{n\geq 5} ef_{n-1} \cdot x^n + \sum_{n\geq 5} ef_{n-2} \cdot x^n =$$
$$= 2 \cdot x^3 + 4 \cdot x^4 + x \cdot \sum_{n\geq 3} ef_n \cdot x^n - ef_3 \cdot x^4 + x^2 \cdot \sum_{n\geq 3} ef_n \cdot x^n$$
$$s(x) = 2 \cdot x^3 + 4 \cdot x^4 + x \cdot s(x) - 2 \cdot x^4 + x^2 \cdot s(x) \Rightarrow s(x) = \sum_{n\geq 3} ef_n \cdot x^n = \frac{2(x^4 + x^3)}{1 - x - x^2}.$$

For $k = 1$, we denote the Extended Fibonacci Cube $EFC_1(n)$ as $EFC_n, n \geq 3$. The cube polynomial of a graph is the counting polynomial for the number of induced k-dimensional hypercubes (subgraphs of G which are H_k). For a graph G, let $c_n(G), n \geq 0$ be the number of induced subgraphs of G isomorphic to H_n. In particular, we have that $c_0(G) = |V(G)|, c_1(G) = |E(G)|$ and $c_2(G)$ is the number of the 4-cycles in G. The cube polynomial $C(G, x)$ of G is the corresponding counting polynomial, that is the generating function $C(G, x) = \sum_{n\geq 0} c_n(G) \cdot x^n$.

In [1] Brešar et al. showed that $C(H_p, x) = (x+2)^p$. We have that $EFC_3 = H_1, EFC_4 = H_2$ and therefore $C(EFC_3, x) = x + 2, C(EFC_4, x) = (x + 2)^2$. From Fig. 1 we see that

$$C(EFC_5) = 6 + 7x + 2x^2 = (x + 2)(2x + 3),$$
$$C(EFC_6) = 10 + 15x + 7x^2 + x^3 = (x + 2)(x^2 + 5x + 5).$$

□

Proposition 2.2. *The generating function of* $\{C(EFC_n, x)\}_{n\geq 3}$ *is*

$$\sum_{n\geq 3} C(EFC_n, x) \cdot y^n = \frac{(x + 2) \cdot y^3(xy + y + 1)}{1 - y - y^2(1 + x)}.$$

Proof. The Extended Fibonacci Cube EFC_n can be decomposed in two Extended Fibonacci Cubes EFC_{n-1} and EFC_{n-2}, each node in EFC_{n-2} is connected to a single node in EFC_{n-1}. This means that $C(EFC_n, x) = C(EFC_{n-1}, x) + (1 + x) \cdot C(EFC_{n-2}, x), n \geq 5$. Then we have that

$$s(x, y) = \sum_{n\geq 3} C(EFC_n, x) \cdot y^n =$$
$$= C(EFC_3, x) \cdot y^3 + C(EFC_4, x) \cdot y^4 + \sum_{n\geq 5} C\big(EFC_{n-1}, x) + (1+x) \cdot C(EFC_{n-2}, x)\big) \cdot y^n =$$
$$= (x + 2) \cdot y^3 + (x + 2)^2 \cdot y^4 + \sum_{n\geq 5} \big(C(EFC_{n-1}, x) + (1 + x) \cdot C(EFC_{n-2}, x)\big) \cdot y^n =$$
$$= (x + 2) \cdot y^3 + (x + 2)^2 \cdot y^4 + y \cdot \sum_{n\geq 3} C(EFC_n, x) \cdot y^n -$$
$$-C(EFC_3) \cdot y^4 + y^2(1 + x) \cdot \sum_{n\geq 3} C(EFC_n, x) \cdot y^n =$$
$$= (x + 2) \cdot y^3 + (x + 2)^2 \cdot y^4 + y \cdot s(x, y) - (x + 2) \cdot y^4 + y^2(1 + x) \cdot s(x, y)$$

and it follows that

$$s(x,y)=\frac{(x+2)\cdot y^3(xy+y+1)}{1-y-y^2(1+x)}$$

is the generating function of $\{C(EFC_n,x)\}_{n\geq 3}$. $\square$

Using the same technique, for any $k\geq 1$ we obtain

Proposition 2.3. *The generating function of* $\{C(EFC_k(n),x)\}_{n\geq k+2}$ *is*

$$\sum_{n\geq k+2} C(EFC_k(n),x)y^n=\frac{(x+2)^k\cdot y^{k+2}(1+y+xy)}{1-y-y^2(1+x)}.$$

Theorem 2.1. *For any* $n\geq 3$, $C(EFC_n,x)$ *is of degree* $\lfloor\frac{n}{2}\rfloor$.

Proof. We have that $\frac{1}{1-x}=\sum_{m\geq 0}x^m$ then it follows that

$$\begin{aligned}
s(x,y)&=\frac{(x+2)\cdot y^3(xy+y+1)}{1-y-y^2(1+x)}=\\
&=(x+2)\cdot y^3(xy+y+1)\cdot\sum_{m\geq 0}(y+y^2(1+x))^m=\\
&=(x+2)\cdot y^3(xy+y+1)\cdot\sum_{m\geq 0}y^m(1+y(1+x))^m=\\
&=(x+2)\cdot y^3\cdot\sum_{n\geq 0}y^m(1+y(1+x))^{m+1}=\\
&=(x+2)\cdot y^3\cdot\sum_{n\geq 0}y^m\cdot\sum_{k=0}^{m+1}\binom{m+1}{k}(y(1+x))^k=\\
&=(x+2)\cdot y^3\cdot\sum_{n\geq 0}y^m\cdot\sum_{k=0}^{m+1}\binom{m+1}{k}\cdot y^k\cdot\sum_{i=0}^{k}\binom{k}{i}\cdot x^i=\\
&=(x+2)\cdot\sum_{m\geq 0}\sum_{k=0}^{m+1}\sum_{i=0}^{k}\binom{m+1}{k}\cdot\binom{k}{i}\cdot y^{m+k+3}\cdot x^i=\\
&=(x+2)\cdot\sum_{m\geq 0}\sum_{n=m+3}^{2m+4}\sum_{i=0}^{n-m-3}\binom{m+1}{n-m-3}\cdot\binom{n-m-3}{i}\cdot y^n\cdot x^i=\\
&=\sum_{m\geq 0}\sum_{n=m+3}^{2m+4}\sum_{i=0}^{n-m-3}\binom{m+1}{n-m-3}\cdot\binom{n-m-3}{i}\cdot x^i\cdot(x+2)\cdot y^n=\\
&=\sum_{n=m+3}^{2m+4}\left[\sum_{m\geq 0}\sum_{i=0}^{n-m-3}\binom{m+1}{n-m-3}\cdot\binom{n-m-3}{i}\cdot x^i\cdot(x+2)\right]\cdot y^n
\end{aligned}$$

On the other hand, $s(x,y) = \sum_{n\geq 3} C(EFC_n, x) \cdot y^n$, so we obtain

$$C(EFC_n, x) = \sum_{m\geq 0} \sum_{i=0}^{n-m-3} \binom{m+1}{n-m-3} \cdot \binom{n-m-3}{i} \cdot x^i \cdot (x+2) =$$
$$= (x+2) \cdot \sum_{m\geq 0} \binom{m+1}{n-m-3} \cdot (1+x)^{n-m-3} =$$
$$= (x+2) \cdot \sum_{m=0}^{n-3} \binom{m+1}{n-m-3} \cdot (1+x)^{n-m-3}$$

But, we have that $m+3 \leq n \leq 2m+4$ and thus $\frac{n}{2}-2 \leq m \leq n-3 \Leftrightarrow 0 \leq n-m-3 \leq \frac{n}{2}-1$.

We obtain then that $C(EFC_n, x)$ is of degree $1 + \lfloor \frac{n}{2} \rfloor - 1 = \lfloor \frac{n}{2} \rfloor,\ n \geq 3$ and

$$C(EFC_n, x) = (x+2) \cdot \sum_{m=0}^{n-3} \binom{m+1}{n-m-3} \cdot (1+x)^{n-m-3} = (x+2) \cdot \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1} \binom{n-k-2}{k} \cdot (1+x)^k.$$

□

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