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About the radius of convexity of some analytic functions

OLGA ENGEL[1,2], PÁL A. KUPÁN[1,2] and ÁGNES O. PÁLL- SZABÓ[1,2]

ABSTRACT. In this paper we prove a general result regarding the radius of convexity for different particular functions. The method of convolutions is used. The results are applied to deduce sharp bounds regarding functions, which satisfy differential subordinations.

1. INTRODUCTION

Let $U(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$ denote the disk of radius r and center z_0. We denote $U(r) = U(0, r)$, and $U = U(1)$. The radius of convergence of the power series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

will be denoted by r_f.
For $r \in (0, r_f)$ we say that the function f is convex in the disk $U(r) = \{z \in C : |z| < r\}$ if f is univalent in $U(r)$, and $f(U(r))$ is a convex domain in $\mathbb{C}$. A function f of the form (1.1) is convex if and only if

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \ \ z \in U(r).$$

In [2] the radius of convexity of the function f is defined by the equality

$$r_f^c = \sup\left\{r \in (0, r_f) : \ \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \ \ z \in U(r)\right\}.$$

The first works which deals with the question of the radius of starlikeness of particular(Bessel) functions were [5] and [8]. The radius of convexity of Bessel functions were determined in [2] at the first time. The radius of starlikeness and the radius of convexity of special functions have been determined in the papers [1], [2], [3], [4], [5], [6], [12]. Other results regarding the starlikness of Bessel functions are given in [13]. Bounds for analytic functions, which satisfy a differential inequality are given in [14] and [15].

2. PRELIMINARIES

We denote by $\mathcal{H}(U)$ the class of analytic functions defined in U.
Let A_0 and $\mathcal{P}$ be the sets of functions defined by

$$A_0 = \{f \in \mathcal{H}(U) |\ f(0) = 1\} \ \text{ and } \ \mathcal{P} = \{f \in A_0 |\ \mathrm{Re}\, f(z) > 0, \ z \in U\}.$$

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Corresponding author: Engel Olga; engel_olga@hotmail.com

Lemma 2.1. [7] *p. 27 Herglotz.*
The function f belongs to the class $\mathcal{P}$ if and only if there is a probability measure μ on $[0, 2\pi]$ such that

$$f(z) = \int_0^{2\pi} \frac{1+e^{-it}z}{1-e^{-it}z} d\mu(t) = 1 + 2\sum_{n=1}^{\infty} z^n \int_0^{2\pi} e^{-int} d\mu(t).$$

As we saw in the abstract we will apply the convolution theory in the study of the convexity of analytic functions. A basic work in the field of convolution is [9]. Recall the following definitions and results from this book.

Let f and g be two analytic functions defined by the power series $f(z) = \sum\limits_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum\limits_{n=1}^{\infty} b_n z^n$. The Hadamard product of these functions is defined by

$$(f * g)(z) = \sum_{n=1}^{\infty} a_n b_n z^n.$$

For $V \subset A_0$ the dual set of V is defined by

$$V^d = \{g \in A_0 | (f * g)(z) \neq 0, \text{ for all } f \in V \text{ and for all } z \in U\}.$$

Lemma 2.2. [9] *Let the function h_T be defined by the power series $h_T(z) = z + \sum\limits_{n=2}^{\infty} \frac{n+iT}{1+iT} z^n$.*
The function $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ is starlike in U if and only if

$$\frac{f(z)}{z} * \frac{h_T(z)}{z} \neq 0, \textit{ for all } z \in U, \textit{ and for all } T \in \mathbb{R}.$$

Lemma 2.3. [6] *Let the class $\mathcal{L}$ be defined by*

$$\mathcal{L} = \{f \in A_0 : \operatorname{Re} f(z) > \frac{1}{2}, \ z \in U\}.$$

The following inclusion holds $\mathcal{L} \subset \mathcal{P}^d$.

3. Main results

Theorem 3.1. *Let f be an analytic function in U given by the equality*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{3.2}$$

If

$$\sum_{n=2}^{\infty} n^2 |a_n| \leq 1, \tag{3.3}$$

then the function f is convex in U.

Proof. The function f is convex if and only if $g(z) = zf'(z)$ is starlike. According to Lemma 2 the function g is starlike in U if and only if

$$\frac{g(z)}{z} * \frac{h_T(z)}{z} \neq 0, \text{ for all } z \in U, \text{ and for all } T \in \mathbb{R}. \tag{3.4}$$

$$\frac{g(z)}{z} * \frac{h_T(z)}{z} = 1 + \sum_{n=1}^{\infty}(n+1)a_{n+1}\frac{n+1+iT}{1+iT}z^n$$
$$= \Big(1 + 2\sum_{n=1}^{\infty} z^n\Big) * \Big(1 + \frac{1}{2}\sum_{n=1}^{\infty}(n+1)a_{n+1}\frac{n+1+iT}{1+iT}z^n\Big). \tag{3.5}$$

According to Lemma 3 the starlikeness condition (3.4) holds if

$$\mathrm{Re}\Big(1 + \frac{1}{2}\sum_{n=1}^{\infty}(n+1)a_{n+1}\frac{n+1+iT}{1+iT}z^n\Big) > \frac{1}{2}, \ \ z \in U.$$

This condition is equivalent to

$$\mathrm{Re}\Big(1 + \sum_{n=1}^{\infty}(n+1)a_{n+1}\frac{n+1+iT}{1+iT}z^n\Big) > 0, \ \ z \in U.$$

We put $z = r(\cos\theta + i\sin\theta), \ r \in (0,1)$, and we get

$$\mathrm{Re}\Big(1 + \sum_{n=1}^{\infty}(n+1)a_{n+1}\frac{n+1+iT}{1+iT}z^n\Big) = \frac{1}{1+T^2}\Big[T^2\Big(1$$
$$+\sum_{n=1}^{\infty}(n+1)a_{n+1}r^n\cos n\theta\Big) + T\Big(\sum_{n=1}^{\infty}(n+1)na_{n+1}r^n\sin n\theta\Big)$$
$$+1 + \sum_{n=1}^{\infty}(n+1)^2a_{n+1}r^n\cos n\theta\Big]. \tag{3.6}$$

We have

$$1 + \sum_{n=1}^{\infty}(n+1)a_{n+1}r^n\cos n\theta \geq 1 - \sum_{n=1}^{\infty}(n+1)|a_{n+1}|r^n >$$
$$1 - \sum_{n=1}^{\infty}(n+1)^2|a_{n+1}|r^n \geq 0, \ \ r \in (0,1), \ \theta \in [0, 2\pi]. \tag{3.7}$$

Thus the starlikeness condition (3.4) holds if

$$\Delta_T(\theta) = \Big(\sum_{n=1}^{\infty}(n+1)na_{n+1}r^n\sin n\theta\Big)^2 - 4\Big(1 + \sum_{n=1}^{\infty}(n+1)a_{n+1}r^n\cos n\theta\Big)$$
$$\Big(1 + \sum_{n=1}^{\infty}(n+1)^2a_{n+1}r^n\cos n\theta\Big) < 0, \ \ \theta \in \mathbb{R}, \ r \in (0,1). \tag{3.8}$$

On the other hand we have

$$\Delta_T(\theta) \leq 4\Big(\sum_{n=1}^{\infty}(n+1)na_{n+1}r^n\sin\frac{n\theta}{2}\cos\frac{n\theta}{2}\Big)^2 - 4\Big(1 - \sum_{n=1}^{\infty}(n+1)|a_{n+1}|r^n\Big)$$
$$\Big(1 + \sum_{n=1}^{\infty}(n+1)^2a_{n+1}r^n\cos n\theta\Big), \ \ \theta \in \mathbb{R}, \ r \in (0,1). \tag{3.9}$$

Now we replace the number 1 in the last two brackets by the smaller expression

$$1+\sum_{n=1}^{\infty}(n+1)^2|a_{n+1}|$$

and we get

$$\Delta_T(\theta)\le 4\Big(\sum_{n=1}^{\infty}(n+1)na_{n+1}r^n\sin\frac{n\theta}{2}\cos\frac{n\theta}{2}\Big)^2-8\Big(\sum_{n=1}^{\infty}n^2|a_{n+1}|r^n\Big)$$
$$\Big(\sum_{n=1}^{\infty}(n+1)^2|a_{n+1}|r^n\frac{1\pm\cos n\theta}{2}\Big),\ \ \theta\in\mathbb{R},\ r\in(0,1). \tag{3.10}$$

We will prove that

$$\Big(\sum_{n=1}^{\infty}(n+1)na_{n+1}r^n\sin\frac{n\theta}{2}\cos\frac{n\theta}{2}\Big)^2-\Big(\sum_{n=1}^{\infty}n^2|a_{n+1}|r^n\Big)$$
$$\Big(\sum_{n=1}^{\infty}(n+1)^2|a_{n+1}|r^n\frac{1\pm\cos n\theta}{2}\Big)\le 0,\ \ \theta\in\mathbb{R},\ r\in(0,1). \tag{3.11}$$

The inequality (3.11) holds because according to the Cauchy-Schwarz inequality, we have

$$\Big(\sum_{n=1}^{\infty}(n+1)n|a_{n+1}|r^n\sqrt{\frac{1\pm\cos n\theta}{2}}\Big)^2\le\Big(\sum_{n=1}^{\infty}n^2|a_{n+1}|r^n\Big)$$
$$\Big(\sum_{n=1}^{\infty}(n+1)^2|a_{n+1}|r^n\frac{1\pm\cos n\theta}{2}\Big)\le 0,\ \ \theta\in\mathbb{R},\ r\in(0,1),$$

and it follows that

$$\Big(\sum_{n=1}^{\infty}(n+1)na_{n+1}r^n\sin\frac{n\theta}{2}\cos\frac{n\theta}{2}\Big)^2\le\Big(\sum_{n=1}^{\infty}(n+1)n|a_{n+1}|r^n\sqrt{\frac{1\pm\cos n\theta}{2}}\Big)^2\le$$
$$\Big(\sum_{n=1}^{\infty}n^2|a_{n+1}|r^n\Big)\Big(\sum_{n=1}^{\infty}(n+1)^2|a_{n+1}|r^n\frac{1\pm\cos n\theta}{2}\Big)\le 0,\ \ \theta\in\mathbb{R},\ r\in(0,1).$$

Finally (3.10) and (3.11) imply $\Delta_T(\theta)\le 0,\ \ \theta\in[0,2\pi],\ \ T\in\mathbb{R}$, and the proof is done. □

Remark 3.1. As far as we know the method of convolution is applied here at the first time in the study of the convexity of analytic functions.

Theorem 3.2. *Let f be an analytic function in U, of the form $f(z)=\sum\limits_{n=1}^{\infty}b_nz^n$, and*

$$\mathrm{Re}\Big(1+\alpha_0 f(z)+\alpha_1 zf'(z)+\ldots+\alpha_p z^p f^{(p)}(z)\Big)>0,\ z\in U. \tag{3.12}$$

We denote $P(n)=\alpha_0+\sum\limits_{k=1}^{p}\alpha_k n(n-1)\ldots(n-k+1)$. If $r\in(0,1]$ and

$$\sum_{n=2}^{\infty}n^2\Big|\frac{P(1)}{P(n)}\Big|\le 1, \tag{3.13}$$

then

$$2\sum_{n=1}^{\infty}\frac{1}{P(n)}r^n>\mathrm{Re}\,f(z)>2\sum_{n=1}^{\infty}\frac{(-1)^n}{P(n)}r^n,\ \ z\in U(r).$$

The result is sharp.

Proof. Let $b_1 = 1$, and $f(z) = \sum\limits_{n=1}^{\infty} b_n z^n$. We have

$$1 + \sum_{k=0}^{p} z^k \alpha_k f^{(k)}(z) = 1 + \alpha_0 f(z) + \sum_{k=1}^{p} \alpha_k \sum_{n=1}^{\infty} n(n-1)\dots(n-k+1) b_n z^n$$
$$= 1 + \sum_{n=1}^{\infty} \Big(\alpha_0 + \sum_{k=1}^{p} \alpha_k n(n-1)\dots(n-k+1)\Big) b_n z^n = 1 + \sum_{n=1}^{\infty} P(n) b_n z^n. \tag{3.14}$$

The Herglotz formula, (3.12), and (3.14) imply

$$1 + \sum_{n=1}^{\infty} P(n) b_n z^n = 1 + 2 \sum_{n=1}^{\infty} z^n \int_0^{2\pi} e^{-tin} d\mu(t),$$

and we get $b_n = \frac{2}{P(n)} \int_0^{2\pi} e^{-tin} d\mu(t)$. Thus it follows that

$$f(z) = \sum_{n=1}^{\infty} \frac{2z^n}{P(n)} \int_0^{2\pi} e^{-tin} d\mu(t). \tag{3.15}$$

According to Theorem 1 the condition (3.13) implies that the function

$$g(z) = z + \sum_{n=2}^{\infty} \frac{P(1)}{P(n)} z^n$$

is convex in U. This implies that

$$f^*(z) = 2 \sum_{n=1}^{\infty} \frac{z^n}{P(n)}$$

is also convex. The equality (3.15) can be rewritten as follows

$$f(z) = \int_0^{2\pi} f^*(ze^{-ti}) d\mu(t), \tag{3.16}$$

which means that $f(z)$ can be written as a convex combination of the values of f^*. The convexity of f^* and the equality (3.16) imply $f(U) \subset f^*(U)$, and this inclusion implies the subordination $f \prec f^*$. The subordination and the convexity of f^* imply $f^*(r) > \operatorname{Re} f(z) \geq f^*(-r),\ z \in U(r)$ which is equivalent to

$$2 \sum_{n=1}^{\infty} \frac{1}{P(n)} r^n > \operatorname{Re} f(z) > 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{P(n)} r^n, \quad z \in U(r),$$

and the proof is finished. □

Remark 3.2. The functions

$$f_t(z) = \sum_{n=1}^{\infty} \frac{2z^n}{P(n)} e^{-tin}, \ t \in [0, 2\pi], \quad z \in U$$

are the extreme points of the class determined by the differential inequality (3.12). It is easily seen that the radius of convergence of these functions are equal to one. Thus the radius of convexity in case of each extreme functions are equal to one.

Corollary 3.1. *Let f be an analytic function in U of the form $f(z)=\sum\limits_{n=1}^{\infty} b_n z^n$. If*

$$\mathrm{Re}\Big(1+f(z)+f'(z)+7z^2f''(z)+6z^3f^{(3)}(z)+z^4f^{(4)}(z)\Big)>0,\ z\in U, \tag{3.17}$$

then for every $z\in U$, the following inequality holds

$$\mathrm{Re}\, f(z)>2\sum_{n\geq 1}\frac{(-1)^n}{n^4+1}=-1+\frac{i}{2}\Big[\frac{\pi}{\overline{\zeta}}\Big(e^{-\pi\overline{\zeta}}+\frac{2\pi\overline{\zeta}}{e^{2\pi\overline{\zeta}}-1}\Big)-\frac{\pi}{\zeta}\Big(e^{-\pi\zeta}+\frac{2\pi\zeta}{e^{2\zeta}-1}\Big)\Big],$$

where $\zeta=\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}$, and $\overline{\zeta}=\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}$.

Proof. We have

$$P(n)=1+n+7n(n-1)+6n(n-1)(n-2)+n(n-1)(n-2)(n-3)=n^4+1$$

It is easily seen that $\sum\limits_{n\geq 2}\frac{P(1)}{P(n)}<\sum\limits_{n\geq 2}\frac{2}{n^4}=2\Big(\frac{\pi^4}{90}-1\Big)\leq\frac{2}{3}$. According to Theorem 2, if $\sum\limits_{n\geq 2}\frac{P(1)}{P(n)}\leq 1$, then it follows that

$$\mathrm{Re}\, f(z)>2\sum_{n=1}^{\infty}\frac{(-1)^n}{P(n)}=2\sum_{n\geq 1}\frac{(-1)^n}{n^4+1},\ z\in U.$$

In order to finish the proof, we will use the identity

$$\sum_{n=-\infty}^{\infty}\frac{e^{in\alpha}}{n^2+\zeta^2}=\frac{\pi}{\zeta}\Big(e^{-\zeta|\alpha|}+\frac{2\alpha\zeta}{e^{2\pi\zeta}-1}\Big),\ -2\pi<\alpha<2\pi,\ \mathrm{Re}\,\zeta>0.$$

Putting in this equality $\alpha=\pi$, $\zeta=\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}$, and $\overline{\zeta}=\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}$, and subtracting the obtained two equalities we obtain

$$2\sum_{n\geq 1}\frac{(-1)^n}{n^4+1}=-1+\frac{i}{2}\Big[\frac{\pi}{\overline{\zeta}}\Big(e^{-\pi\overline{\zeta}}+\frac{2\pi\overline{\zeta}}{e^{2\pi\overline{\zeta}}-1}\Big)-\frac{\pi}{\zeta}\Big(e^{-\pi\zeta}+\frac{2\pi\zeta}{e^{2\pi\zeta}-1}\Big)\Big].$$

□

Corollary 3.2. *Let g be the analytic function defined by*

$$g_n(z)=\frac{(n+1)!}{z^n}\Big(e^z-1-\frac{z}{1!}-\frac{z^2}{2!}-\ldots-\frac{z^n}{n!}\Big). \tag{3.18}$$

If $n\geq 7$ then g_n is convex in U.

Proof. The function g_n is of the form (3.2). Thus in order to prove the convexity of g_n it is enough to check the condition (3.3).
Since

$$g_n(z)=z+\sum_{p=2}^{\infty}\frac{z^p}{(n+2)\ldots(n+p)}.$$

it follows that the mapping g_n is convex provided that

$$\alpha_n=\sum_{p=2}^{\infty}\frac{p^2}{(n+2)\ldots(n+p)}\leq 1. \tag{3.19}$$

Since the sequence $(\alpha_n)_{n\geq 1}$ is strictly decreasing it is enough to prove (3.19) in case $n=7$. If $n \geq 7$, then we have

$$\alpha_n \leq \alpha_7 < \sum_{p=2}^{\infty} \frac{p^2}{9^{p-1}} = \frac{298}{512} = 0.58\ldots \leq 1,$$

and the proof is done. □

Remark 3.3. Since the function $g_n, \quad n \geq 7$ maps the unit disk U on a convex domain $g_n(U)$, which is simmetric with respect to the real axys, the following double inequality holds

$$g_n(-1) \leq \operatorname{Re} g_n(e^{i\theta}) \leq g_n(1), \quad \theta \in [0, 2\pi].$$

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[1]BABEŞ - BOLYAI UNIVERSITY
DEPARTMENT OF MATHEMATICS
[2] SAPIENTIA HUNGARIAN UNIVERSITY OF TRANSYLVANIA
DEPARTMENT OF MATHEMATICS-INFORMATICS
E-mail address: engel_olga@hotmail.com
E-mail address: kicsim21@yahoo.com
E-mail address: kupanp@ms.sapientia.ro