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Mathematical excursions in fractal world

Anna Soós

Abstract. In this article we will rewiev the existence and uniqueness results for fractal sets. The fractals are sets having noninteger Hausdorff dimensions. In selfsimilar case this dimension can be calculated using the similarity dimension.

"Mathematics, rightly viewed, possesses not only truth, but supreme beauty"
– Bertrand Russell, from The Study of Mathematics: Philosophical Essays

What is a fractal?

B. Mandelbrot:

A rough or fragmented geometric shape that can be subdivided in parts, each of which is (at least approximately) a reduced/size copy of the whole.

Mathematical: A set of points whose fractal dimension is noninteger.

Traditionally, a line is thought of as 1-dimensional object; a plane as a 2-dimensional object and a prism as a 3-dimensional object. Dimensions are seen as having integer values. The term 'fractal' suggests the ideea that some objects have a 'fractional' dimension. In this article we will take an excursion in so called "fractal geometry". Mandelbrot's fractal geometry provides a mathematical model for many complex forms found is nature such as shapes of coast lines, mountains, galaxy clusters, and clouds.

1. Basic notions

Let X be a nonempty set and d a metric on X. The classical example is the Euclidian space $\mathbb{R}^n$ with the Euclidian metric $d(x,y)=\sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}$.

Let $f: X \to X$ and let $x_0 \in X$. Define

$$x_1 = f(x_0)$$

$$x_2 = f(x_1) = f \circ f(x_0) = f^2(x_0)$$

$$\ldots\ldots$$

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$$x_n = f(x_{n-1}) = \underbrace{f \circ f \circ \ldots \circ f}_{n \text{ times.}}(x_0) = f^n(x_0).$$

Let (X, d_X) and (Y, d_Y) be metric spaces. A map $\varphi : X \to Y$ is said to be **Lipschitz** on X if

$$d_Y(\varphi(x), \varphi(y)) \leq r d_X(x, y)$$

for all $x, y \in X$, where r is a positive number called the **Lipschitz constant** of φ.

The **iterated function system** (IFS) consists of a family of contractions $\mathbf{S} := \{\varphi_1, ..., \varphi_m\}$ on X. If there exists a set K such that

$$K = \cup_{i=1}^{u} \varphi_i(K),$$

it is called the **invariant set** of the IFS.

Let (X, d) be a metric space. If $\varphi : X \to X$ is Lipschitz on X and the Lipschitz constant is less then 1, then f is called a **contraction** with respect to the metric d with contractivity ratio r. In particular, a contraction φ with contraction ratio r is called a **similitude** if $d(\varphi(x), \varphi(y)) = rd(x, y)$ for all $x, y \in X$.

It is known from the calssical analysis the Banach's contraction principle

Theorem 1.1. *Let (X, d) be a complete metric space and let $\varphi : X \to X$ be a contraction with respect to the metric d. Then there exists a unique fixed point of φ, in other words, there exists a unique solution to the equation $\varphi(x) = x$. Moreover, if x_* is the fixed point of φ, then $\{\varphi^n(a)\}_{n \geq 0}$ converges to x_* for all $a \in X$ where φ^n is the n-th iteration of φ.*

If A is a subset of X and $r > 0$, then the r **neighbourhood** of A is

$$A_r := \{y : \ d(x, y) < r \text{ for some } x \in A\}.$$

Let $\mathcal{C}(X)$ the class of nonempty compact subsets of X.

The **Hausdorff metric** on $\mathcal{C}(X)$ is defined as

$$h(A, B) := inf\{r : \ A \subseteq B_r \text{ and } \ B \subseteq A_r\}.$$

On can show that the Hausdorff metric is a metric on $\mathcal{C}(X)$ and if (X, d) is a complete metric space then $(\mathcal{C}(X), h)$ is also complete.

Now we can prove, following [4], the existence and uniqueness of fractals:

Theorem 1.2. *Let (X, d) be a complete metric space and let $\varphi_i : X \to X$ be a contraction for $i \in \{1, 2, ..., m\}$, $m \in \mathbb{N}$. Define $\mathbf{S} : \mathcal{C}(X) \to \mathcal{C}(X)$ by*

$$\mathbf{S}(A) := \cup_{i=1}^{m} \varphi_i(A).$$

Then $\mathbf{S}$ has a unique fixed point K. Moreover, for any $A \in \mathcal{C}(X)$, $\mathbf{S}^n(A)$ converges to K as $n \to \infty$ with respect to the Hausdorff metric, where $\mathbf{S}^n$ is *the n-th iterate of* $\mathbf{S}$ *and*

$$h(\mathbf{S}^n(A), K) \leq \frac{r^n}{1-r} h(A, \mathbf{S}(A)) \to 0$$

as $n \to \infty$*. Furthermore, if* $A \in \mathcal{C}(X)$ *is such that* $\varphi_i(A) \subset A$ *for all* i*, then*

$$K = \cap_{i=0}^{\infty} \mathbf{S}^i(A).$$

Proof. If $A, B \in \mathcal{C}(X)$ then

$$\begin{aligned} h(\mathbf{S}(A), \mathbf{S}(B)) &= h(\cup_{i=1}^m \varphi_i(A), \cup_{i=1}^m \varphi_i(B)) \leq \\ &\leq \max_{1 \leq i \leq m} h(\varphi_i(A), \varphi_i(B)), \end{aligned}$$

using the definition of metric h and noting that if the ϵ-neighbourhood $(\varphi_i(A))_\epsilon$ contains $\varphi_i(B)$ for all i then $(\cup_{i=1}^m \varphi_i(A))_\epsilon$ contains $\cup_{i=1}^m \varphi_i(B)$ and vice versa. By the definition of contraction

$$h(\mathbf{S}(A), \mathbf{S}(B)) \leq (\max_{1 \leq i \leq m} r_i) h(A, B). \tag{1.1}$$

Since $\max_{1 \leq i \leq m} r_i < 1$, the mapping $\mathbf{S}$ is a contraction on the complete metric space $(\mathcal{C}(X), h)$. By the Banach's contraction principle $\mathbf{S}$ has a unique fixed point and moreover $\mathbf{S}^n(A) \to K$ as $n \to \infty$. By iterating (1.1) it follows that

$$h(\mathbf{S}^n(A), K) \leq (\max_{1 \leq i \leq m} r_i)^n h(A, K).$$

Thus $\mathbf{S}^n(A)$ converges to K at a geometric rate. In particular, if $\varphi_i(A) \subset A$ for all i, then $\mathbf{S}(A) \subset A$, so that $\mathbf{S}^n(A)$ is a decreasing sequence of non-empty compact sets containing K with intersection $\cap_{i=0}^{\infty} \mathbf{S}^i(A)$ which must equal K. □

This unique fixed point $K \subset X$ is **the invariant set of the IFS**. Usually it is a fractal.

2. Selfsimilar fractal sets

If the contractions are similarities, the attractor K is called **selfsimilar**, if they are affine transformations, then K is called **selfaffine**. These sets are frequently fractals.

Example 1. The middle-third Cantor set:

$\varphi_1, \varphi_2 : \mathbb{R} \to \mathbb{R}$*:*

$$\varphi_1(x) = \frac{1}{3}x \qquad \varphi_2(x) = \frac{1}{3}x + \frac{2}{3}$$

FIGURE 1. The triadic Cantor dust

Example 2. The Sierpinski gasket

Let q_1, q_2, q_3 *the vertices of an equilateral tiangle.*

$\varphi_i : \mathbb{R}^2 \to \mathbb{R}^2$

$$\varphi_i(x) = \frac{1}{2}(x - q_i) + q_i,\ i = 1, 2, 3.$$

FIGURE 2. The Sierpinski gasket

Example 3. The Menger sponge

Begin with a cub of side 1. Subdivide it into 27 smaller cubes by trisecting the edges. We will remove the center cub and the 6 cubes in the center of the faces. That means 20 cubes remain. Continue in the same way with the small cubes.

Computationally, it is very easy to reconstruct the invariant set K of a given IFS. Let I_k the set of all k-term sequences $(i_1, ..., i_k)$ with $i_j \in \{1, ..., n\}$. Plotting $\mathbf{S}^k(A) = \cup_{I_k} \varphi_{i_1}(A) \circ \varphi_{i_2}(A) \circ \cdots \circ \varphi_{i_k}(A)$ for a suitable k gives an approximation to K. (See Figure 4.)

An alternative way of reconstructing K is to take any initial point x_0, and select a sequence $\varphi_{i_1}, \varphi_{i_2}, ...$ independently at random from the given contractions. Then the points defined by

$$x_k = \varphi_{i_k}(x_{k-1}), \text{ for } k = 1, 2...$$

are indistinguishably close to K.

Better results will be obtained by weighting the probabilities of choosing the φ_i. (See Figure 5.)

Figure 3. The Menger sponge

Figure 4. The Sierpinski gasket

Figure 5. The Sierpinski gasket

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3. HAUSDORFF MEASURE AND DIMENSION

In this section we will introduce the notion of the Hausdorff measure and dimension, and we will show how to calculate the Hausdorff dimension of selfsimilar sets.

Let (X, d) a metric space and $A \subset X$ be a bounded subset. Let

$$\mathcal{H}^s_\delta(A) = \inf \left\{ \sum_{i \geq 1} diam(E_i)^s \,|\, A \subset \bigcup_{i \geq 1} E_i,\, diam(E_i) \leq \delta \right\},$$

where the infimum is taken over all δ-covers of A. Also, we define

$$\mathcal{H}^s(A) = \limsup_{\delta \to 0} \mathcal{H}^s_\delta(A).$$

$\mathcal{H}^s(A)$ is called the **s-dimensional Hausdorff measure of A** .

It is well-known that $\mathcal{H}^s$ is a complete Borel regular measure for any $s > 0$.

Theorem 3.1. [2] *Let (X, d) be a metric space. For any $A \subset X$ we have*

$$\sup\{s \,|\, \mathcal{H}^s(A) = \infty\} = \inf\{s \,|\, \mathcal{H}^s(A) = 0\}. \tag{3.2}$$

Proof. First we show, for $0 \leq s < t$,

$$\mathcal{H}^t_\delta(A) \leq \mathcal{H}^{t-s}_\delta(A) \tag{3.3}$$

for any $A \subseteq X$. For, let $A \subseteq \bigcup_{i \geq 1} E_i$ and $diam(E_i) \leq \delta$ for any i, then

$$\begin{aligned} \sum_{i \geq 1} diam(E_i)^t &\leq \sum_{i \geq 1} diam(E_i)^{t-s} diam(E_i)^s \leq \\ &\leq \delta^{t-s} \sum_{i \geq 1} diam(E_i)^s \end{aligned}$$

By the inequality (3.3), if $s < t$, then $\mathcal{H}^s(A) < \infty$ implies $\mathcal{H}^t(A) = 0$ and also $\mathcal{H}^s(A) = \infty$. ☐

Using this theorem we can give the notion of Hausdorff dimension.

The quantity given by the equality (3.2) is called the **Hausdorff dimension of A**, which is denoted by $dim_H(A)$.

The Hausdorff measure and the Hausdorff dimension depend on the metric d.

Mandelbrot named **fractal** the set having noninteger Hausdorff dimension.

The first result concerning the Hausdorff dimension of selfsimilar fractal sets is essentially due to Moran. In [4] Moran's theorem and proof are presented inthe language of iterated function system. In order to obtain the formula of Hausdorff dimension of selfsimilar sets, one has to impose the open set condition.

Assume that K is the invariant set of the IFS $\mathbf{S} = (\varphi_i, i = 1, ..., m)$ where φ_i are similitudes with contractivities $r_i \in [0, 1)$, $i = 1, ..., m$. The IFS satisfies the **open set condition** if there exists a nonempty bounded open set $G \subset X$ such that

$$\bigcup_{i=1}^{m} \varphi_i(G) \subseteq G$$

and $\varphi_i(G) \cap \varphi_j(G) = \emptyset$ for $i \neq j$.

Theorem 3.2. [3] *Suppose that* K *is the invariant set of the IFS* $(X, (\varphi_i \,| i = 1, ..., m))$ *and the open set condition is satisfied. Then*

$$dim_H K \;=\; D,$$

where D *is the unique positive solution of*

$$\sum_{i=1}^{m} r_i^D \;=\; 1. \tag{3.4}$$

Proof. Let $n \in \mathbb{N}$. Since K is the invariant set, we have

$$K = \bigcup_{i=1}^{m} \varphi_i(K).$$

This implies that

$$K = \bigcup_{i(n) \in \Sigma_n} \varphi_{i(n)}(K),$$

where $\Sigma = \{1, ..., m\}^{\mathbb{N}}$, $\Sigma_n = \{1, ..., m\}^{\{1,...,n\}}$ and $i(n) = (i_0, ..., i_n)$, $i_j \in \{1, ..., m\}$. Since the composition of the similitudes $\varphi_{i(n)}$ is a similitude with contractivity $r_{i(n)} := r_{i_0} ... r_{i_n}$, (3.4) implies that

$$\begin{aligned} \sum_{i(n) \in \Sigma_n} (diam \varphi_{i(n)}(K))^D &= \sum_{i(n) \in \Sigma_n} (r_{i(n)})^D (diam K)^D = \\ &= (\sum_{i_1} r_{i_1}^D) ... (\sum_{i_n} r_{i_n}^D)(diam K)^D = (diam K)^D. \end{aligned}$$

Now given any $\epsilon > 0$, one can always find an $n \in \mathbb{N}$ large enough so that

$$diam \varphi_{i(n)}(K)^D \leq (\max_i r_i)^D \leq \epsilon.$$

Thus

$$\mathcal{H}_\epsilon^D(K) \leq (diam K)^D,$$

and consequently

$$\mathcal{H}^D(K) \leq (diam K)^D.$$

To obtain a lower bound a measure ν on n-cylinders $Z_{i(n)} := \{i \in \Sigma |\, i = i(n) j\}$ is introduced.

Define

$$\nu Z_{i(n)} := (r_{i(n)})^D.$$

It follows from (3.4) that

$$\nu Z_{i(n)} = \sum_i \nu Z_{i(n)i}$$

and therefore $\nu\Sigma = 1$. This ν can be extended to a measure $\overline{\nu}$ on K.

Let G the nonempty bounded open set whose existence is guaranteed by the open set condition. The fact that every compact set converges to the attractor K implies

$$\overline{K} \supseteq \mathbf{S}(\overline{K}) \supseteq ... \supseteq \mathbf{S}^n(\overline{K}) \to K.$$

Therefore

$$\varphi_{i(n)}\overline{G} \subseteq \varphi_{i(n)}(K),$$

for all $i(n)$, $n \in \mathbb{N}$.

Now let B be a ball of radius $r < 1$ intersecting K. Let $i \in \Sigma$ and let n be the first integer for which

$$(\min_i r_i)r \leq r_{i(n)} \leq r.$$

Denote by Σ^* the set of all such strings. For any $i \in \Sigma$ there exists exactly one integer n such that $i(n) \in \Sigma^*$. Since $\{\varphi_1(G), ..., \varphi_m(G)\}$ is disjoint, so is $\{\varphi_{i(n)1}(G), ..., \varphi_{i(n)m}(G)\}$, for all $i(n) \in \Sigma_n$. Hence, the collection $\{\varphi_{i(n)}(G) | i(n) \in \Sigma^*\}$ is disjoint, and therefore

$$K \subseteq \bigcup_{i(n)\in\Sigma^*} \varphi_{i(n)}(K) \subseteq \bigcup_{i(n)\in\Sigma^*} \varphi_{i(n)}(\overline{G}).$$

Now choose two real numbers ρ_1 and ρ_2 such that G contains a ball of radius $\rho_1 r$ and is contained in a ball of radius $\rho_2 r$. If $i(n) \in \Sigma^*$, the set $\varphi_{i(n)}(G)$ contains a ball of radius $r_{i(n)}\rho_1$ and thus one of radius $(\min_i r_i)\rho_1 r$ and is contained in a ball of radius $r_{i(n)}\rho_2$ and hence in one of radius $\rho_2 r$. Now denote Σ^{**} the set of all codes in Σ^* for which $\varphi_{i(n)}(G) \cap B \neq \emptyset$. Denote by m the number of sets $\varphi_i(\overline{G})$ that intersect B. The sum over the volumes of the interior balls yields

$$m(\rho_1 r)^n \leq (1 + 2\rho_2)^n r^n.$$

Then there are at most $m = (1 + 2\rho_2)^n \rho_1^{-1} (\min_i r_i)^{-n}$ codes in Σ^{**}..

Then

$$\overline{\nu}B = \overline{\nu}B \cap K \leq \nu(\bigcup_{i(n)\in\Sigma^{**}} Z_{i(n)}).$$

Thus

$$\begin{aligned} \overline{\nu}B &\leq \sum_{i(n)\in\Sigma^{**}} \nu Z_{i(n)} = \sum_{i(n)\in\Sigma^{**}} \nu r_{i(n)}^D \leq \\ &\leq \sum_{i(n)\in\Sigma^{**}} r^D \leq m r^D. \end{aligned}$$

As any set U is contained in a ball of radius $diamU$, $\overline{\nu}U \leq m(diamU)^D$. So $\mathcal{H}^D(K) \geq m^{-1} > 0$, and thus $dim_H(K) = D$. $\square$

The similarity dimension can be used to compute the Hausdorff dimension when the two coincide. For example the Cantor dust has similarity dimension $\log 2/\log 3$, the Sierpinski gasket $\log 3/\log 2$ and the Menger sponge $\log 20/\log 3$.

4. FRACTAL MEASURE

It is usually more convenient to work with measures rather than sets. For applications such as image compression it is convenient to consider greyscales.

Let (X, d) be a complete separable metric space.

A **probabilistic iterated function system** is a $2m$-tuple

$$\mathbf{S} := (p_1, \varphi_1, ..., p_m, \varphi_m),\ m \geq 1,$$

of positive real numbers p_i such that $\sum_{i=1}^{m} p_i = 1$ and of Lipschitz maps $\varphi_i : X \to X$. Let r_i the Lipschitz constants of φ_i, $i \in \{1, ..., m\}$.

Denote $M = M(X)$ the set of finite mass Radon measures on X with the weak topology. If $\mu \in M$, then the measure $\mathbf{S}\mu$ is defined by

$$\mathbf{S}\mu = \sum_{i=1}^{m} p_i \varphi_i \mu,$$

where $\varphi_i\mu$ is the usual push forward measure, i.e.

$$\varphi_i\mu(A) = \mu(\varphi_i^{-1}(A)), \text{ for } A \subseteq X.$$

We say μ **is an invariant measure** if $\mathbf{S}\mu = \mu$.

If the contractions are similarities, then μ is called **selfsimilar fractal measure.** Let M_q denote the set of unit mass Radon measures μ on X with finite q-th moment. That is,

$$M_q = \{\mu \in M |\, \mu(X) = 1,\ \int_X d^q(x, a) d\mu(x) < \infty\}$$

for some (and hence any) $a \in X$. Note that, if $p \geq q$ then $M_p \subset M_q$.

The l_q **minimal metric** l_q on M_q is defined by

$$l_q(\mu, \nu) = \inf\{(\int_X d^q(x, y) d\gamma(x, y))^{\frac{1}{q} \wedge 1} |\, \pi_1\gamma = \mu,\ \pi_2\gamma = \nu\}$$

where $\wedge$ denotes the minimum of the relevant numbers and $\pi_i\gamma$ denotes the i-th marginal of γ, i.e. projection of the measure γ on $X \times X$ onto the i-th component.

The following theorem was proved in [4] in case $q = 1$ and in general in [6].

Theorem 4.1. *If* $\mathbf{S} = (p_1, \varphi_1, \cdots, p_m, \varphi_m)$ *is a probabilisitc IFS and*

$$\lambda_q := \sum_{i=1}^{m} p_i r_i^q < 1$$

for some $q > 0$*, then there is a unique invariant measure* $\mu^* \in M_q$ *of* $\mathbf{S}$*. Moreover, for any* $\mu_0 \in M_q$,

$$l_q(\mathbf{S}^k \mu_0, \mu^*) \leq \frac{\lambda_q^{k(\frac{1}{q}\wedge 1)}}{1 - \lambda_q^{\frac{1}{q}\wedge 1}} l_q(\mu_0, \mathbf{S}\mu_0) \to 0 \text{ as } k \to \infty.$$

Proof. We have $\mathbf{S} : M_q \to M_q$. Moreover,

$$\begin{aligned} l_q^{q\vee 1}(\mathbf{S}\mu, \mathbf{S}\nu) &= l_q^{q\vee 1}(\sum_{i=1}^{m} p_i \varphi_i \mu, \sum_{i=1}^{m} p_i \varphi_i \nu) \leq \\ &\leq \sum_{i=1}^{m} p_i l_q^{q\vee 1}(\varphi_i \mu, \varphi_i \nu) \leq \sum_{i=1}^{m} p_i r_i^q l_q^{q\vee 1}(\mu, \nu) \end{aligned}$$

from the properties of l_q. Hence $\mathbf{S}$ is a contraction map with contraction constant $\lambda_q^{\frac{1}{q}\wedge 1}$. This implies the theorem □

Let

$$M_0 := \cup_{q>0} M_q.$$

Since

$$(\int_X \log d^q(x, a) d\mu(x))^{\frac{1}{q}} \to \exp \int_X \log d(x, a) d\mu(x)$$

as $q \to 0$, it follows that

$$M_0 = \{\mu \in M \,|\, \mu(X) = 1, \int_X \log d(x, a) d\mu(x) < \infty\}.$$

Since $\lambda_q^{\frac{1}{q}} \to \prod_{i=1}^{m} r_i^{p_i}$ as $q \to 0$, it follows that if $\prod_{i=1}^{m} r_i^{p_i} < 1$ (i.e. $\sum_{i=1}^{m} p_i \log r_i < 0$), then there is a unique measure $\mu^* \in M_0$ which satisfies $\mathbf{S}$. Moreover, for any $\mu_0 \in M_0$, $\mathbf{S}^k \mu_0 \to \mu^*$ in the weak sense of measures as $k \to \infty$.

It also follows that the μ^* in the theorem is unique in the M_0.

Since $\lambda_q^{\frac{1}{q}} \to \max_{1 \leq i \leq N} r_i$ as $q \to \infty$, then the support of μ denoted by $spt\mu^*$ is compact and is the unique invariant compact set of the IFS $(\varphi_1, ..., \varphi_m)$. Moreover, if $spt\mu_0$ is compact then $spt\mathbf{S}^k\mu_0 \to spt\mu^*$ in the Hausdorff metric sense.

There is a random algorithm for constructing the invariant measure μ. Let $(i_1, i_2, ...)$ be a random sequence such that $i_j = i$ with probability p_i, independently for each j. Fixing $x \in spt\mu$, we define for each Borel set A

$$\mu_x(A) = \lim_{k\to\infty} \frac{1}{k} card\{k' \leq k \,\text{such that}\; \varphi_{i_{k'}} \circ \cdots \circ \varphi_{i_1}(x) \in A\}.$$

Then for μ-almost all x we have $\mu_x(A) = \mu(A)$. Thus iterating x under a random sequence of mappings with S_i chosen with probability p_i, the proportion of iterates lying in a set A approximates $\mu(A)$.

For example taking $p_1 = p_2 = \frac{1}{2}$ and

$$\varphi_1(x) = \frac{1}{3}x \qquad \varphi_2(x) = \frac{1}{3}x + \frac{2}{3}$$

gives the so called Cantor measure. The Figure 6 is an example based on the Menger sponge.

Figure 6. The probabilistic Menger sponge

References

[1] Barnsley M.F., *Fractals Everywhere*, Academic Press, 1993

[2] Edgar G.A., *Measure, Topology, and Fractal Geometry*, Springer, 1990

[3] Falconer K.J., *Techniques in fractal geometry,* John Wiley & Sons, 1997

[4] Hutchinson J.E., *Fractals and Self Similarity,* Indiana University Mathematics Journal, 30 (1981), no. 5, 713-747

[5] Hutchinson J.E., Rüschendorf J.E., *Random Fractal Measures via the Contraction Method,* Indiana Univ. Math. Journal, 47(2), (1998), 471-489

[6] Hutchinson J.E., Rüschendorf L., *Selfsimilar Fractals and Selfsimilar Random Fractals,* Progress in Probability, 46, (2000), 109-123

[7] Kolumbán J., Soós Anna, *Invariant sets of random variables in complete metric spaces,* Studia Univ. "Babes-Bolyai", Mathematica, XLVII, 3 (2001), 49-66

[8] Mandelbrot B., *How long is the coast of Britain? Statistical self similarity and fractional dimension*, Science, 155, (1967), 636-638

[9] Mandelbrot B., *The Fractal Geometrie of Nature*, W. H. Freeman and Company, New York, 1977

BABEŞ-BOLYAI UNIVERSITY
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
M. KOGĂLNICEANU 1, 400084 CLUJ-NAPOCA, ROMANIA
E-mail address: `asoos@math.ubbcluj.ro`