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Some methods for composing problems in mathematics

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Abstract. The article sustains the idea that the mathematics education should be performed as a continuous research and discovery, not just as a simple transmission of already known ideas. An essential contribution to this activity would be the invention of new mathematical problems. Here are some methods: paraphrasing, changing of statement's data, analogy, generalization, combination.

The contemporary school has a well determined role in developing of a huge intellectual potential, represented by intelligence and creativity, that, being capitalized properly, can provide an uninterrupted social-human progress. In the relation between pupil and learning process, knowing the level of intellectual advancement of each student is very important for the utilization of adequate methods, that would allow the individualization of the education, so as each pupil would promote at the most his creative capacities and abilities.

From the point of view of the creativity it is necessary the revision of the traditional teaching methods used in school, by assimilation of creative strategies and promotion of new methods.

At the second Congress for the mathematical education (Ecster City, 1972), the famous French mathematician R.Tom considered that it was necessary to pass to the "creative" heuristic method of teaching and learning. Also, a lot of European mathematician scientists pleaded for a new conception of mathematical instruction (1982), according to what:

- mathematics is to be considered as a learning activity and not a finished study object;
- mathematics is to be studied with interest and not by imposed memorization;
- mathematical instruction is to be produced as a research and permanent discovery and not as a simple conveyance of already known ideas.

The systematic solving of mathematical issues contributes to the conscientious learning of knowledge and, especially to the accumulation by pupils of experience that would involve creative activities, and to the developing of creative skills.

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On the correlation line between creativity and solving mathematical problems we were interested in some aspects:

- pupils' training in creative activities needs a system of cognitive issues, with a researcher character;
- the solving of problems constitutes a favorable frame for creativity development;
- defining for creativity is the part of problems' composition (wording) and not that of their solving (G. Pólya, 1965; A.N. Kolmogorov, 1970; P.C. Wason, J. Laird, 1986; J.T. Dillon, 1988 etc., [1],[2]).

For the composition of mathematical issues we should possess general see also procedures and concrete methods, that would guide our actions in the direction of achieving the wanted result.

The general procedures of mathematical problems' composing can be the following:

- The establishment of experimental statements, by as much as possible numerical verifications (possible truths);
- The making of helping constructions in geometry, that are wiped up after statement's wording;
- Considering a problem (theorem), to formulate another one (theorem) using elements of mathematical logic;
- Considering a problem (theorem), to formulate a new one (theorem) using generalizations, analogies etc.
- The establishment of connections between different domains (combinations);
- The solving of a known problem, using another method.

The composing of mathematical issues activity should start with their controlled composing. Then, applying methods as paraphrasing, changing of statement's data, analogy, combination, that has an advanced degree of independence, we can obtain original problems.

Directed mathematical problems' composing. The achievement of new knowledge by self-instruction is a creative process. An essential help in this direction is constituted of creative tasks, namely the directed mathematical problems' composing. The last means the issues' composing activity initiated and unfolded under permanent teacher's supervision. In this meaning the mechanism of demonstration problems' composing could be described in the following way:

- the selection of objects and objectives for research (teacher);
- the analysis of problem's situation (pupil);
- the obtaining of new information about the object (pupil);
- problem's statement's wording based on the ascertained fact (pupil);
- the making evident the applied theme in problem's solving (pupil-teacher);
- the obtaining formulated problem's solution (pupil-teacher);
- conclusions (pupil-teacher).

The analysis of problem's situation depends on the selected objects and objectives, and it could be realized:

- on the basis of calculus, constructions and measures (induction);
- on the basis of deduction of logical consequences from the selected conditions (deduction);
- on the basis of passing information from some object to another one (analogy, generalization).

We do not exclude the existence of other modalities.

The mechanism of demonstration problems' composing determines the technology of organizing pupils' activity for tasks' achievement, that should contain a situation of problem and its object of research.

We shall describe concise the technique of demonstration problem's composing on the basis of calculus, constructions and measures (the inductive method of obtaining knowledge).

The teacher proposes to the pupils a task that contains objects and objectives for study (research). Then every pupil makes the operations for the objects given, corresponding to the objectives announced (at geometry, for example, construct and measure). The results obtained are written in a general table whose analysis will permit the observation of the regularity and the formulation of an hypothesis. The next step of work is the formulation of the problems statement, emphasizing the applied theme for solving and the problem's solution.

Paraphrasing. By paraphrasing we understand a reformulation of the problem. We distinguish the next tips of paraphrasing:
the paraphrasing that does not change the essence of the problem;
the paraphrasing that changes the problem partially;
the paraphrasing by which we obtain a new problem;

The paraphrasing can be made by:
reformulation of the problem in another aspect;
the change of a figure with another(in geometry);
reformulation of the problem in another domain of mathematics (or in another activity domain described in the statement);
reformulation of the problem, applying elements of mathematical logic.

We don't exclude the existence of other tips.

Example (*the author is the known Russian mathematician I.M. Ghelifand*), [3]. *A pawn is situated in a corner of a chess board. Two players move the pawn one by one (in the figure 1 these moves are drawn with arrows). Wins the player that will place the pawn on the opposite field of the board. Who wins in a fair play?*

This is a reformulation of this problem in another tip of activity:

In two clusters are placed by seven stones. Two players make one by one an operation: take a stone from a cluster, or take a stone from each cluster, or moves a stone from a cluster in another. Wins the player that takes the last stone Can the player that starts the game win it?

In order to formulate problems using elements of mathematical logic we can apply the properties:

1) $(p_1 \wedge p_2 \wedge \ldots \wedge p_{i-1} \wedge p_i \wedge p_{i+1} \wedge \ldots \wedge p_n) \rightarrow q \equiv (p_1 \wedge p_2 \wedge \ldots \wedge p_{i-1} \wedge (\rceil q) \wedge p_{i+1} \wedge \ldots \wedge p_n) \rightarrow \rceil p_i$.

2) $(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \rightarrow (q_1 \vee q_2) \equiv (p_1 \wedge p_2 \wedge \ldots \wedge p_n \wedge (\rceil q_1)) \rightarrow q_2$.

3) $(p_1 \wedge \ldots \wedge p_i \wedge \ldots \wedge p_n) \rightarrow q \equiv (p_1 \wedge \ldots \wedge p_{i-1} \wedge p_{i+1} \wedge \ldots \wedge p_n) \rightarrow (q \vee (\rceil p_i))$.

Change of the data in the statement. Among the possibilities of changing the data in the statement we enumerate:

- Replacing one notion with another;
- Replacing several notions with other ones;
- Replacing one relationship from the statement;
- Addition of some conditions;
- Changing numerical values of a constant;
- Combining some of the previous ones.

We suggest there are also other possibilities.

In the result, the problem can be changed partially or a new one can be obtained.

Replacing one definition with another one and replacing several definitions with other ones is made through concretization and specialization. Replacing one relationship from the statement, for example, an equality with an inequality is possible.

The addition of some restrictive conditions is made through particularization (concretization). The change of a numerical value of a constant can lead to generalization.

Example *A quadrilateral regulate pyramid is given. Find the volume, if the height is equal to h, and the angle between a lateral edge and the basis plane is α.*

Replacing the term "quadrilateral" with "triangle" or "hexagon" etc.; the term "height" with "lateral edge" or "apothem" or "basis side" or "radius of the circumcised sphere" etc; angle α with angle β, or γ, or δ, we obtain a series of new problems (β – the dihedral angle between a side and the basis plane, γ – the plane angle from the peak of the pyramid, δ – the dihedral angle, whose edge is a lateral edge).

The analogy. Mathematics has its own particularities of analogy application. These particularities differ from a department to other. Geometry offers a very vast field of activity in which new affirmations can be obtained through analogy. A simple example: the analogy between the geometry of the triangle and the geometry of the tetrahedron.

A simple mechanism of creation of geometrical affirmations (planimetry – stereometry) through analogy can be:

- Selecting an affirmation from planimetry;
- Making a replacement of notions after a well-determined scheme:

Point – straight line, straight line – plane, angle between two straights – dihedral angle between two planes, the length of the segment – the area of the polygonal surface, the area of the polygonal surface – the volume of the polyhedron etc.

- We settle the structure of the affirmation from the stereometry which is analogous with the selected affirmation;
- The obtained hypothesis can be confirmed or infirmed by reasoning;
- If possible, the hypothesis is completed until it becomes a true affirmation.

The theory of polynomial divisibility is an example in algebra of an analogy for the theory of integer numbers‘ divisibility. It would be wrong if this analogy is not taken into consideration by teachers when teaching this material.

We can apply the following method in algebra for composing new mathematical problems:

- We select a problem that has a solution that is obtained from important affirmations;
- If possible, we settle an analogous affirmation to the affirmation from the solution;
- A new mathematical problem is composed using the found affirmation.

The confirmation through reasoning of the solution of the new mathematical problem is evident because both affirmations are based on the same method of construction.

The mathematical analysis is not an exception and contains many examples of analogies. For example: the limit of a function in a certain point and the derivative of a function in a certain point. If we replace some expressions in remarkable limits by other expressions analogical to their limits, we can obtain interesting results. Applying also the combination we can obtain original mathematical problems. We can use the same method when calculating the indefinite integral by applying the method of changing the variable.

It can be concluded that an analogy applied efficiently can lead to a considerable contribution in education.

Generalization. In the list of the methods of generalization of mathematical affirmations we can include :

- The replacement of numbers with parameter;
- The suspension of some restrictions from a mathematical problem;
- The application of the affirmation from a problem to a larger number of objects;
- The transference of a geometrical feature from one object to another;

This list can be enlarged for sure.

The generalization starts from a thorough analysis of the demonstration in order to realize which conditions were essential. This method does not always lead to a result : it can happen that a demonstration does not suggest a generalization, but another demonstration, with another idea, more comprehensive, can suggest it. There are situations when a concrete example serves as a result of a demonstration of a general case.

For example : *The side AD is the diameter of the circumscribed circle to the convex quadrilateral $ABCD$. The point of intersection of the bisectors of angles B and C is situated on the segment AD. Show that $AB + CD = AD$.*

The condition that AD is the diameter of the circle can be omitted. A generalization of the problem [4]:

ABCD is an inscribable quadrilateral and the point of intersection of the bisectors of angles B and C is situated on the segment AD. Demonstrate that $AB + CD = AD$.

The comparison, the abstracting and the induction influence directly the value of the final generalization.

Combination. Combination leads to new mathematical problems.

This method can be realized by logical combination of :

- Two or more results from the same domain;
- Two or more results from different domains.

For example, combining two important results from the theory of symmetrical functions, the theory of algebraic equations and from planimetry (triangle), we can obtain identities, important and original inequalities for the sides of the triangle, for its heights etc. Interesting problems can be obtained by combining important results from geometry planimetry and stereometry and from the theory of inequalities.

A successful combination needs a hard work, a strong intuition and a deep knowledge of the theory.

A practical base of this method is the resolving of such mathematical problems, paying attention to their structure and method of combination. In this case it is necessary to explain thoroughly the scheme of the studied problem.

The list of the methods of composing mathematical problems does not end here. This activity contains more methods.

Here is an example for the last method. Our object of study will be the symmetrical polynomials with three unknown variables and some sizes of the triangle: three sides, or three heights, or the radii of escribed circles etc. A polynomial with three unknown variables x, y, z is called symmetrical if its form does not change at every permutation of its unknown variables. For example :

$$P(x, y, z) = (x - y)^2 + (y - z)^2 + (z - x)^2$$

is a symmetrical polynomial, $Q(x, y, z) = x - y - z$ is not a symmetrical polynomial. The most simple symmetrical polynomials are considered $P_1 = x + y + z$, $P_2 = xy + yz + zx$, $P_3 = xyz$. It can be shown that every symmetrical polynomial with the unknown variables x,y,z can be expressed using the polynomials P_1, P_2, P_3. For example :

$$x^2 + y^2 + z^2 = (P_1)^2 - 2P_2,\ x^3 + y^3 + z^3 = (P_1)^3 - 3P_1P_2 + 3P_3,$$
$$(x + y)(y + z)(z + x) = P_1P_2 - P_3,$$

etc.

The Vietes' formulas for the third degree equations : if x_1, x_2, x_3 are the solutions for the third degree equation

$$a_0x^3 + a_1x^2 + a_2x + a_3 = 0 (a_0 \neq 0)$$

then

$$x_1 + x_2 + x_3 = -a_1/a_0, x_1x_2 + x_2x_3 + x_3x_1 = a_2/a_0, x_1x_2x_3 = -a_3/a_0$$

.

Are given a, b and c the lengths of the sides of the triangle ABC. It can be proved that these values are the solutions of the equation [5]:

$$x^3 - 2\mathrm{px}^2 + (p^2 + r^2 + 4\mathrm{Rr})x - 4\mathrm{pRr} = 0.$$

Applying Vietes' formulas we obtain identities for values of a, b and c:

$$a + b + c = 2p, ab + bc + ca = p^2 + r^2 + 4Rr, abc = 4pRr.$$

From the last relation we obtain a known formula for the area of the triangle : $S = (abc)/4R$. If we continue we obtain:

$$a^2 + b^2 + c^2 = 2(p^2 - r^2 - 4Rr$$
$$a^3 + b^3 + c^3 = 2p(p^2 - 3r^2 - 6Rr)$$
$$(a + b)(b + c)(c + a) = 2p(p^2 + r^2 + 2Rr).$$

We combined the results of the symmetrical polynomials‘ theory with those of the algebraic equations‘ domain and the domain of the planimetry.

The composure of mathematical problems has a considerable contribution in developing the creativity. It helps applying such mental capacities as observation, comparison, description, analysis, synthesis, induction, deduction, analogy and such imagination proceeding as substitution, modification, adaptation etc. If formed these capacities improve the divergent and the convergent way of thinking, the abilities for mathematics and the motivation for studying.

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