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On the convergence of a sequence generated by an integral

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Abstract. Some results concerning the convergence of some sequences generated by integrals are presented. Immediate applications are also given.

1. Introduction

The aim of this note is to present some results which give sufficient conditions for the convergence of some sequences determined by integrals. The following vell known problem is an example for the applicability of our results.

Problem 1.1. *Let $f : [0, 1] \to R$ a continuous function. Prove that*

$$\lim_{n\to\infty} \int_0^1 \frac{f(x^n)}{1+x} dx = f(0) \cdot \ln 2.$$

We need the following known result, whose proof is included for the completeness.

Lemma 1.1. *Let $f : [c, d] \to \mathbb{R}$ be a differentiable function with bounded derivative on $[c, d]$. There exists a sequence of polynomial functions $(P_m)_{m\geq 1}$ such that*
a) $P_m \to f$ uniformly on $[c, d]$, that is $\lim_{m\to\infty} \sup_{t\in[c,d]} |P_m(t) - f(t)| = 0$;
b) $(P'_m)_{m\geq 1}$ is a sequence of equally bounded functions on $[c, d]$, that is $\exists\ A > 0$ such that $\sup_{t\in[c,d]} |P'_m(t)| \leq A$, $\forall m \geq 1$.

Proof. Assume that $[c, d] = [0, 1]$. Let $B_n f$ be the Bernstein polynomial of order n associated to the function f (see [1, 2])

$$B_n f(x) = \sum_{k=0}^{n} C_n^k x^k (1-x)^{n-k} f\left(\frac{k}{n}\right).$$

Denote $\sup_{t\in[0,1]} |f'(t)| = A$. It is known that $B_n f \to f$ uniformly on $[0, 1]$, for any continuous function $f : [0, 1] \to \mathbb{R}$. Moreover, for any differentiable function with bounded derivative on $[0, 1]$, $((B_n f)')_{n\geq 1}$ is a sequence of equally bounded functions on

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$[0,1]$. Indeed,

$$\begin{aligned}
(B_n f)'(x) &= \sum_{k=1}^{n} k C_n^k x^{k-1} (1-x)^{n-k} f\left(\frac{k}{n}\right) \\
&\quad - \sum_{k=0}^{n-1} (n-k) C_n^k x^k (1-x)^{n-k-1} f\left(\frac{k}{n}\right) \\
&= n \sum_{k=1}^{n} C_{n-1}^{k-1} x^{k-1} (1-x)^{n-k} f\left(\frac{k}{n}\right) \\
&\quad - n \sum_{k=0}^{n-1} C_{n-1}^{k} x^k (1-x)^{n-k-1} f\left(\frac{k}{n}\right) \\
&= n \sum_{k=0}^{n-1} C_{n-1}^{k} x^k (1-x)^{n-k-1} \left[f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)\right].
\end{aligned}$$

On the other hand, there exists $c_{nk} \in \left(\frac{k}{n}, \frac{k+1}{n}\right)$ such that:

$$n\left[f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)\right] = n \cdot f'(c_{nk}) \left(\frac{k+1}{n} - \frac{k}{n}\right) = f'(c_{nk})$$

Therefore, for any $n \geq 1$

$$\begin{aligned}
|(B_n f)'(x)| &\leq \sum_{k=0}^{n-1} C_{n-1}^{k} x^k (1-x)^{n-k-1} \left| n\left[f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)\right]\right| \\
&= \sum_{k=0}^{n-1} C_{n-1}^{k} x^k (1-x)^{n-k-1} |f'(c_{nk})| \\
&\leq A \cdot \sum_{k=0}^{n-1} C_{n-1}^{k} x^k (1-x)^{n-k-1} = A \cdot (1-x+x)^{n-1} = A.
\end{aligned}$$

It follows that we can consider $P_m(x) = B_m(x)$, $\forall x \in [0,1]$, and in the general case, $P_m(x) = B_m\left(\frac{x-c}{d-c}\right)$, $\forall x \in [c,d]$. □

2. Main results

Our main results are given in the following three Propositions.

Proposition 2.1. *Let $f : [c,d] \to \mathbb{R}$ be a continuous function, $\lambda_n : [a,b] \to [c,d]$, $n \in \mathbb{N}$ a sequence of continuous functions, and let $g : [a,b] \to \mathbb{R}$ a Riemann integrable function. Assume that there exists $x_0 \in [a,b]$ such that the sequence $(\lambda_n(x_0))_{n\geq 1}$ converges, and the sequence*

$$\alpha_n := \int_a^b |\lambda_n(x) - \lambda_n(x_0)|\, dx,$$

converges to 0. *Then*

$$\lim_{n\to\infty}\int_a^b f\left(\lambda_n\left(x\right)\right)\cdot g\left(x\right)dx = f\left(\lim_{n\to\infty}\lambda_n\left(x_0\right)\right)\cdot\int_a^b g\left(x\right)dx.$$

Proof. If $\lim_{n\to\infty}\lambda_n\left(x_0\right)$ is denoted by l, then $l\in[c,d]$. Since g is Riemann integrable it follows that $|g|$ is also Riemann integrable and g is bounded. Denote $M=\sup_{x\in[a,b]}|g(x)|$. Assume that f is a Lipschitz function, namely there exists $L>0$ such that

$$|f(x)-f(y)|\le L\,|x-y|\,,\;\forall x,y\in[c,d]\,.$$

We have the following estimates

$$\begin{aligned}
&\left|\int_a^b f\left(\lambda_n\left(x\right)\right)\cdot g\left(x\right)dx-\int_a^b f\left(\lambda_n\left(x_0\right)\right)\cdot g\left(x\right)dx\right| \qquad (1)\\
\le\;& \int_a^b\left|f\left(\lambda_n\left(x\right)\right)-f\left(\lambda_n\left(x_0\right)\right)\right|\cdot\left|g\left(x\right)\right|dx\\
\le\;& L\int_a^b\left|\lambda_n\left(x\right)-\lambda_n\left(x_0\right)\right|\cdot\left|g\left(x\right)\right|dx\\
\le\;& ML\cdot\alpha_n,
\end{aligned}$$

and,

$$\begin{aligned}
&\left|\int_a^b f\left(\lambda_n\left(x_0\right)\right)\cdot g\left(x\right)dx-f(l)\cdot\int_a^b g\left(x\right)dx\right| \qquad (2)\\
\le\;& L\cdot\left|\lambda_n\left(x_0\right)-l\right|\cdot\int_a^b\left|g\left(x\right)\right|dx\\
\le\;& (b-a)ML\cdot\left|\lambda_n\left(x_0\right)-l\right|.
\end{aligned}$$

The relations (1) and (2) and the triangle inequality lead to

$$\begin{aligned}
&\left|\int_a^b f\left(\lambda_n\left(x\right)\right)\cdot g\left(x\right)dx-f(l)\cdot\int_a^b g\left(x\right)dx\right| \qquad (3)\\
\le\;& ML\cdot\left[\alpha_n+(b-a)\cdot\left|\lambda_n\left(x_0\right)-l\right|\right].
\end{aligned}$$

Clearly, the right hand side of (3) converges to 0 when $n\to\infty$.

Thus the statement is proved when f is a Lipschitz function. In the general case, the idea is to approximate the function f with a Lipschitz function, uniformly on $[c,d]$.

Applying the Weierstrass's uniform approximation theorem (see for instance Theorem 7.1 from [1], pp. 88–89 or Theorem 2 from [2], pp. 214-216), it follows that there exists a $L(\varepsilon)$ – Lipschitz function $h:[c,d]\to R$ such that

$$|f(x)-h(x)|<\varepsilon,\;\forall x\in[c,d]\,.$$

It is easily seen that

$$\begin{aligned} & \left| \int_a^b f\left(\lambda_n\left(x\right)\right) \cdot g\left(x\right) dx - \int_a^b h\left(\lambda_n\left(x\right)\right) \cdot g\left(x\right) dx \right| \qquad (4) \\ \leq \quad & \varepsilon \int_a^b \left| g\left(x\right) \right| dx \\ \leq \quad & (b-a)M\varepsilon, \end{aligned}$$

and,

$$\begin{aligned} & \left| f(l) \int_a^b g\left(x\right) dx - h(l) \int_a^b g\left(x\right) dx \right| \qquad (5) \\ < \quad & \varepsilon \left| \int_a^b g\left(x\right) dx \right| \\ \leq \quad & \varepsilon \int_a^b \left| g\left(x\right) \right| dx \\ \leq \quad & (b-a)M\varepsilon. \end{aligned}$$

From (4), (5) and from the inequality (3) applied to the function h we get:

$$\begin{aligned} & \left| \int_a^b f\left(\lambda_n\left(x\right)\right) \cdot g\left(x\right) dx - f(l) \int_a^b g\left(x\right) dx \right| \\ \leq \quad & 2(b-a)M\varepsilon + ML(\varepsilon) \cdot \left[\alpha_n + (b-a) \cdot \left|\lambda_n\left(x_0\right) - l\right|\right]. \end{aligned}$$

By hypothesis, there exists $N(\varepsilon) \in \mathbb{N}$ such that

$$\alpha_n + (b-a) \cdot \left|\lambda_n\left(x_0\right) - l\right| < \frac{\varepsilon}{ML(\varepsilon)}, \; \forall n \geq N(\varepsilon).$$

Therefore

$$\left| \int_a^b f\left(\lambda_n\left(x\right)\right) \cdot g\left(x\right) dx - f(l) \int_a^b g\left(x\right) dx \right| < \varepsilon,$$

for all $n \geq N\left(\frac{\varepsilon}{2(b-a)M+1}\right)$. Thus the proof is completed. □

Remark 2.1. *If it is further assumed that the sequence of continuous functions $(\lambda_n)_{n\in\mathbb{N}}$ converges almost everywhere on $[a,b]$, then Proposition 2.1 is a consequence of Lebesgue dominated convergence theorem (see Theorem 4.7. from* [1]*, p. 140). Moreover, the Riemann integrals from the statement of Proposition 2.1 are equal with the corresponding Lebesgue integrals.*

Proof. Denote $\lim_{n\to\infty} \lambda_n\left(x\right)$ by $\lambda(x)$ whenever this limit exists, i.e. for almost every $x \in [a,b]$, and set $\lambda(x_0) = \lim_{n\to\infty} \lambda_n\left(x_0\right)$. The Lebesgue integrable functions

$$F_n(x) := f\left(\lambda_n\left(x\right)\right) \cdot g\left(x\right),$$

are uniformly bounded on $[a,b]$, since

$$|F_n(x)| \leq M \cdot \sup_{y\in[c,d]} |f(y)| < \infty,$$

and

$$\lim_{n\to\infty} F_n(x) = f\left(\lambda\left(x\right)\right)\cdot g\left(x\right) =: F(x)$$

for almost every $x \in [a,b]$. From the dominated convergence theorem it follows that F is Lebesgue integrable and

$$\lim_{n\to\infty} \int_a^b F_n\left(x\right) d\mu = \int_a^b F\left(x\right) d\mu.$$

Clearly, $|\lambda_n| \leq 2(d-c)$ on $[a,b]$. Applying the dominated convergence theorem to the sequence $\left(\lambda_n\right)_{n\in\mathbb{N}}$ it follows that λ is a Lebesgue integrable function, and

$$\begin{aligned} 0 = \lim_{n\to\infty} \alpha_n &= \lim_{n\to\infty} \int_a^b \left|\lambda_n\left(x\right) - \lambda_n\left(x_0\right)\right| dx \\ &= \int_a^b \lim_{n\to\infty} \left|\lambda_n\left(x\right) - \lambda_n\left(x_0\right)\right| dx \\ &= \int_a^b \left|\lambda\left(x\right) - \lambda\left(x_0\right)\right| dx. \end{aligned}$$

This implies $\lambda\left(x\right) = \lambda\left(x_0\right)$ almost everywhere on $[a,b]$, and

$$\begin{aligned} \int_a^b F\left(x\right) d\mu &= \int_a^b f\left(\lambda\left(x\right)\right)\cdot g\left(x\right) d\mu \\ &= \int_a^b f\left(\lambda\left(x_0\right)\right)\cdot g\left(x\right) d\mu \\ &= f\left(\lambda\left(x_0\right)\right)\cdot \int_a^b g\left(x\right) d\mu. \end{aligned}$$

Therefore $\lim_{n\to\infty} \int_a^b f\left(\lambda_n\left(x\right)\right)\cdot g\left(x\right) dx = f\left(\lambda\left(x_0\right)\right)\cdot \int_a^b g\left(x\right) dx$. □

Using Remark 2.1 we can give an alternative proof of Proposition 2.1. Indeed, the assumption $\lim_{n\to\infty} \alpha_n = 0$ shows that the sequence

$$\mu_n : [a,b] \to \mathbb{R},\ \mu_n(x) := \lambda_n(x) - \lambda_n(x_0),$$

is convergent to zero in $L^1([a,b])$. Then this sequence has a subsequence

$$\left(\mu_{n_k}\right)_{k\in\mathbb{N}}$$

which is convergent to zero almost everywhere on $[a,b]$. According to Remark 2.1, we get

$$\lim_{k\to\infty} \int_a^b f\left(\lambda_{n_k}\left(x\right)\right)\cdot g\left(x\right) dx = f\left(\lim_{n\to\infty} \lambda_n\left(x_0\right)\right)\cdot \int_a^b g\left(x\right) dx.$$

The claim of Proposition 2.1 will be proved if we check that the sequence $\left(\int_a^b f\left(\lambda_n\left(x\right)\right)\cdot g\left(x\right) dx\right)_{n\in\mathbb{N}}$ is Cauchy. Let $\varepsilon > 0$. Using the same notations as in the proof of Proposition 2.1, we see that

$$\begin{aligned} & \left|\int_a^b h\left(\lambda_n(x)\right) g(x) dx - \int_a^b h\left(\lambda_m(x)\right) g(x) dx\right| \\ \leq\ & ML(\varepsilon)\left[\alpha_n + \alpha_m + (b-a)\left|\lambda_n(x_0) - \lambda_m(x_0)\right|\right], \end{aligned}$$

whence the sequence $\left(\int_a^b h\left(\lambda_n\left(x\right)\right)\cdot g\left(x\right)dx\right)_{n\in\mathbb{N}}$ is Cauchy. Furthermore, using (4) we get

$$\left|\int_a^b f\left(\lambda_n(x)\right)g(x)dx-\int_a^b f\left(\lambda_m(x)\right)g(x)dx\right|$$
$$\leq\quad ML(\varepsilon)\left[\alpha_n+\alpha_m+(b-a)\left|\lambda_n(x_0)-\lambda_m(x_0)\right|\right]+2(b-a)M\varepsilon.$$

Since $(\lambda_n(x_0))_{n\in\mathbb{N}}$ is a Cauchy sequence and $\lim_{n\to\infty}\alpha_n=0$, the last inequality shows that the sequence $\left(\int_a^b f\left(\lambda_n\left(x\right)\right)\cdot g\left(x\right)dx\right)_{n\in\mathbb{N}}$ is Cauchy. The proof is now completed. □

Example 2.1. *In the following cases it is known that the condition*

$$\lim_{n\to\infty}\alpha_n=0,$$

holds (with the notations from Proposition 2.1).

(1) $[a,b]=[0,1]$, $\lambda_n(x)=x^n$, $x_0=0$, (or $\lambda_n(x)=(1-x)^n$, $x_0=1$), $[c,d]=[0,1]$;

(2) $[a,b]=[-1,1]$, $\lambda_n(x)=(1-x^2)^n, x_0=1$, $[c,d]=[0,1]$;

(3) $[a,b]=[0,\pi/2]$, $\lambda_n(x)=\sin^n x, x_0=0$, $[c,d]=[0,1]$;

(4) $[a,b]=[0,\pi/2]$, $\lambda_n(x)=\cos^n x, x_0=\pi/2$, $[c,d]=[0,1]$.

Proposition 2.1 implies then, whenever f is continuous on $[c,d]$ and g is Riemann integrable on $[a,b]$

(1) $\lim_{n\to\infty}\int_0^1 f\left(x^n\right)\cdot g\left(x\right)dx=f\left(0\right)\cdot\int_0^1 g\left(x\right)dx$;

(2) $\lim_{n\to\infty}\int_{-1}^1 f\left((1-x^2)^n\right)\cdot g\left(x\right)dx=f\left(0\right)\cdot\int_0^1 g\left(x\right)dx$;

(3) $\lim_{n\to\infty}\int_0^{\pi/2} f\left(\sin^n x\right)\cdot g\left(x\right)dx=f\left(0\right)\cdot\int_0^{\pi/2} g\left(x\right)dx$.

(4) $\lim_{n\to\infty}\int_0^{\pi/2} f\left(\cos^n x\right)\cdot g\left(x\right)dx=f\left(0\right)\cdot\int_0^{\pi/2} g\left(x\right)dx$.

Of course, we can build new examples from the former ones adding to each sequence of functions $(\lambda_n)_{n\geqslant 1}$ a convergent sequence of constants $(c_n)_{n\in\mathbb{N}}$ and assuming now that

$$[c,d]=\left[\inf_{n\in\mathbb{N}}c_n,1+\sup_{n\in\mathbb{N}}c_n\right].$$

A direct application of Proposition 2.1 is the following

Problem 2.1. *Let* $\alpha\in(0,1)$ *and* $I_n(\alpha):=\int_0^\alpha\ln(1+x+...+x^{n-1})dx$ *for every* $n\geq 2$. *Calculate* $\lim_{n\to\infty}I_n(\alpha)$.

Proof: Denote $I = \int\limits_0^{\alpha} \ln(1-x)dx$. We have $I_n(\alpha) + I = \int\limits_0^{\alpha} \ln(1-x^n)dx$. We may apply Proposition 2.1 for $[a,b] = [0,\alpha]$, $[c,d] = [1-\alpha, 1]$, $\lambda_n(x) = 1 - x^n$, $x_0 = 0$, $f(x) = \ln x$ for every $x \in [1-\alpha, 1]$ and $g \equiv 1$. Then $\lim_{n\to\infty}(I_n(\alpha) + I) = 0$, hence $\lim_{n\to\infty} I_n(\alpha) = -I = \alpha - (\alpha - 1)\ln(1-\alpha)$. □

Proposition 2.2. *Let $f : [c,d] \to \mathbb{R}$ be a differentiable function with bounded derivative, $g : [a,b] \to \mathbb{R}$ a Riemann integrable function which admits a primitive, $(\lambda_n)_{n\geq 1}$, $\lambda_n : [a,b] \to [c,d]$, $n \in \mathbb{N}$ a sequence of differentiable functions with integrable derivatives on $[a,b]$. If there exists a sequence of real, nonzero numbers $(\alpha_n)_{n\geq 1}$, such that*

$$\lim_{n\to\infty} \alpha_n \cdot \int_a^b |\lambda_n'(x)|\, dx = 0,$$

and there exists $l \in [c,d]$ such that

$$\lim_{n\to\infty} \alpha_n (\lambda_n(a) - l) = 0,$$

then

$$\lim_{n\to\infty} \alpha_n \int_a^b [f(\lambda_n(x)) - f(l)]\, g(x)dx = 0.$$

Proof. Let G be a primitive of the function g on $[a,b]$. Next we will prove that the sequence $(A_n)_{n\geq 1}$ given by $A_n := \alpha_n \int_a^b [f(\lambda_n(x)) - f(l)]\, g(x)dx$ tends to 0 in the case when for any $n \geq 1$, $f' \circ \lambda_n$ is a Riemann integrable function on $[a,b]$. Then $(f \circ \lambda_n)' = (f' \circ \lambda_n)(\lambda_n)'$ is integrable on $[a,b]$ and

$$\begin{aligned} A_n &= \alpha_n [f(\lambda_n(b)) - f(l)]\, G(b) - \alpha_n [f(\lambda_n(a)) - f(l)]\, G(a) \\ &\quad -\alpha_n \int_a^b f'(\lambda_n(x)) \cdot (\lambda_n)'(x) \cdot G(x)dx \end{aligned}$$

Let G be the primitive of g such that $G(b) = 0$. Then

$$A_n = -B_n - C_n \cdot G(a),$$

where

$$B_n := \alpha_n \int_a^b f'(\lambda_n(x)) \cdot (\lambda_n)'(x) \cdot G(x)dx,$$

and

$$C_n := \alpha_n [f(\lambda_n(a)) - f(l)]\,.$$

Clearly,

$$|B_n| \leq \sup_{t\in[c,d]} |f'(t)| \cdot \sup_{x\in[a,b]} |G(x)| \cdot |\alpha_n| \cdot \int_a^b \left|(\lambda_n)'(x)\right| dx.$$

Therefore $\lim_{n\to\infty} B_n = 0$. If $\lambda_n(a) \neq l$, it follows from Lagrange's theorem that there exists ξ_n between $\lambda_n(a)$ and l, such that

$$f(\lambda_n(a)) - f(l) = f'(\xi_n) \cdot (\lambda_n(a) - l)\,.$$

Thus,

$$|f(\lambda_n(a)) - f(l)| \leq \sup_{t\in[c,d]} |f'(t)| \cdot |\lambda_n(a) - l|\,, \forall n \geq 1,$$

which together with hypothesis leads to $\lim_{n\to\infty} C_n = 0$.

We have proved that if $f' \circ \lambda_n$, $n \geq 1$ are Riemann integrable functions on $[a, b]$, then $\lim_{n\to\infty} A_n = 0$. For the proof of the general case we will use Lemma 1.1. Let P be a polynomial function. Then

$$\begin{aligned} |A_n(f) - A_n(P)| &= \left| \alpha_n \int_a^b [f(\lambda_n(x)) - f(l)]\, g(x)dx \right. \\ &\quad \left. - \alpha_n \int_a^b [P(\lambda_n(x)) - P(l)]\, g(x)dx \right| \\ &\leq |\alpha_n| \cdot \int_a^b |f(\lambda_n(x)) - P(\lambda_n(x))| \cdot |g(x)|\, dx \\ &\quad + |\alpha_n| \cdot \int_a^b |f(l) - P(l)| \cdot |g(x)|\, dx \\ &\leq 2 \cdot |\alpha_n| \cdot \sup_{t\in[c,d]} |f(t) - P(t)| \cdot \int_a^b |g(x)|\, dx. \end{aligned} \tag{6}$$

From the proof of the particular case it follows that:

$$\begin{aligned} |A_n(P)| &\leq \sup_{t\in[c,d]} |P'(t)| \cdot \Big\{ |G(a)| \cdot |\alpha_n(\lambda_n(a) - l)| \\ &\quad + \sup_{x\in[a,b]} |G(x)| \cdot |\alpha_n| \cdot \int_a^b \left| (\lambda_n)'(x) \right| dx \Big\}. \end{aligned} \tag{7}$$

From (6) and (7) we deduce that:

$$\begin{aligned} |A_n(f)| &\leq 2 \cdot |\alpha_n| \cdot \int_a^b |g(x)|\, dx \cdot \sup_{t\in[c,d]} |f(t) - P(t)| \\ &\quad + \sup_{t\in[c,d]} |P'(t)| \cdot \sup_{x\in[a,b]} |G(x)| \cdot \Big\{ |\alpha_n(\lambda_n(a) - l)| \\ &\quad + |\alpha_n| \cdot \int_a^b \left| (\lambda_n)'(x) \right| dx \Big\}. \end{aligned}$$

Let P be one of the polynomials P_m from Lemma 1.1. Then $\sup_{t\in[c,d]} |P'(t)| \leq A$. We notice that it suffices to study the case when $G \neq 0$ and $\int_a^b |g(x)|\, dx \neq 0$, otherwise the statement of the proposition is obvious. Consider $\varepsilon > 0$. Then there exists $N(\varepsilon)$ such that,

$$|\alpha_n(\lambda_n(a) - l)| + |\alpha_n| \cdot \int_a^b \left| (\lambda_n)'(x) \right| dx < \frac{\varepsilon}{2} A \cdot \sup_{x\in[a,b]} |G(x)|,$$

$\forall n \geq N(\varepsilon)$. Therefore, $\forall n \geq N(\varepsilon)$ and $\forall m \geq 1$ it follows

$$|A_n(f)| < 2 \cdot |\alpha_n| \cdot \int_a^b |g(x)|\, dx \cdot \sup_{t\in[c,d]} |f(t) - P_m(t)| + \frac{\varepsilon}{2}. \tag{8}$$

For any n there exists $m = m(n)$ such that

$$\sup_{t\in[c,d]} |f(t) - P_m(t)| < \frac{\varepsilon}{4|\alpha_n| \cdot \int_a^b |g(x)|\,dx}.$$

For all $n \geq N(\varepsilon)$, and for $m = m(n)$ the inequality (8) implies

$$|A_n(f)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, for any $n \geq N(\varepsilon)$ we have $|A_n(f)| < \varepsilon$. This completes the proof. □

Problem 2.2. *Let $f : [0,1] \to \mathbb{R}$ be a differentiable function with bounded derivative, $g : [0,1] \to \mathbb{R}$ a Riemann integrable function which admits a primitive, and let P be a polynomial. Then*

$$\lim_{n\to\infty} \ln(P(n)) \int_0^1 \left[f\left(\frac{\ln(1+nx)}{n}\right) - f(0) \right] g(x)dx = 0.$$

Proof. The statement is a direct consequence of Proposition 2.2 if we choose $l = 0$, $\lambda_n : [0,1] \to [0,1]$, $\lambda_n(x) := \dfrac{\ln(1+nx)}{n}$, $n \in \mathbb{N} \setminus 0$ and the sequence $(\alpha_n)_{n\geq 1}$ be given by $\alpha_n := \ln(P(n))$. □

The last result of this note is the following:

Proposition 2.3. *Let $f : [c,d] \to \mathbb{R}$ be a continuous function , differentiable at some point $y_0 \in [c,d]$, let $g : [a,b] \to \mathbb{R}$ be a Riemann integrable function, continuous at b, with $g(b) = 0$. Let $(\lambda_n)_{n\geq 1}$, $\lambda_n : [a,b] \to [c,d]$, $n \in \mathbb{N}$, be a sequence of continuous functions with $\lambda_n(a) = y_0$ and $\lambda_n(x) \neq y_0$ for $x \neq a$, and let $(\alpha_n)_{n\geq 1}$ be a sequence of real numbers with $\alpha_n > 0$, $\forall n \geq 1$. Assume that the folowing conditions are satisfied*

(1) $\lim_{n\to\infty} \alpha_n \cdot \int_a^\delta |\lambda_n(x) - y_0|\,dx = 0$, $\forall \delta \in (a,b)$;

(2) There exists a positive constant k such that $\forall \delta \in (a,b)$ there exists $n_{0,\delta} \in \mathbb{N}$ such that the inequality

$$\alpha_n \cdot \int_\delta^b |\lambda_n(x) - y_0|\,dx \leq k,$$

holds $\forall n \geq n_{0,\delta}$.

Then

$$\lim_{n\to\infty} \alpha_n \int_a^b (f(\lambda_n(x)) - f(y_0)) \cdot g(x)dx$$
$$= f'(y_0) \cdot \lim_{n\to\infty} \alpha_n \cdot \int_a^b (\lambda_n(x) - y_0) \cdot g(x)dx = 0.$$

Proof. Consider $\varphi : [c,d] \to \mathbb{R}$,

$$\varphi(x) = \begin{cases} \frac{f(y)-f(y_0)}{y-y_0} & , y \in [c,d]\backslash\{y_0\} \\ f'(y_0) & , y = y_0 \end{cases}$$

Notice that φ is a continuous function on $[c,d]$. We have :

$$\alpha_n \cdot \int_a^b (f(\lambda_n(x)) - f(y_0)) \cdot g(x)dx =$$

$$= \alpha_n \int_a^b \varphi(\lambda_n(x)) \cdot (\lambda_n(x) - y_0) \cdot g(x)dx.$$

There exists $m > 0$ such that $|\varphi(y)| \leq m$, $\forall y \in [c,d]$. Denote

$$M = \sup_{x\in[a,b]} g(x).$$

Consider $\varepsilon > 0$. There exists $\delta_0 \in (a,b)$ such that $|g(x)| < \frac{\varepsilon}{2km}$, $\forall x \in [\delta_0, b]$. Consequently, there exists $n_{0,\delta_0} \in \mathbb{N}$ such that for any $n \geq n_{0,\delta_0}$ the relation

$$\left|\alpha_n \int_{\delta_0}^b \varphi(\lambda_n(x)) \cdot (\lambda_n(x) - y_0) \cdot g(x)dx\right| \tag{9}$$

$$\leq \alpha_n \int_{\delta_0}^b |\varphi(\lambda_n(x))| \cdot |\lambda_n(x) - y_0| \cdot |g(x)|\, dx$$

$$\leq m \cdot \frac{\varepsilon}{2km} \cdot \alpha_n \cdot \int_{\delta_0}^b |\lambda_n(x) - y_0|\, dx < \frac{\varepsilon}{2}$$

holds. On the other hand, there exists $n_1 \in \mathbb{N}$ such that for any $n \geq n_1$:

$$\left|\alpha_n \int_a^{\delta_0} \varphi(\lambda_n(x)) \cdot (\lambda_n(x) - y_0) \cdot g(x)dx\right| \tag{10}$$

$$\leq \alpha_n \int_a^{\delta_0} |\varphi(\lambda_n(x))| \cdot |\lambda_n(x) - y_0| \cdot |g(x)|\, dx$$

$$\leq m \cdot M \cdot \alpha_n \int_a^{\delta_0} |\lambda_n(x) - y_0|\, dx < \frac{\varepsilon}{2}$$

Using (9) – (10) it follows that

$$\left|\alpha_n \int_a^b (f(\lambda_n(x)) - f(y_0)) \cdot g(x)dx\right| < \varepsilon,\ \forall n \geq \max(n_{0,\delta_0}, n_1).$$

Therefore the proof is completed. □

Problem 2.3. *Let $f : [0,1] \to \mathbb{R}$ be a continuous function , differentiable at 0, let $g : [0,1] \to \mathbb{R}$ be a Riemann integrable function, continuous at 1, with $g(1) = 0$, and let $(\alpha_n)_{n\geq 1}$ be a sequence of positive numbers such that the sequence $\left(\frac{\alpha_n}{n}\right)_{n\geq 1}$ is bounded. Then*

$$\lim_{n\to\infty} \alpha_n \int_0^1 (f(x^n) - f(0)) \cdot g(x)dx$$

$$= f'(0) \cdot \lim_{n\to\infty} \alpha_n \int_0^1 x^n \cdot g(x)dx = 0.$$

Proof. Clearly, with $y_0 = 0$ and $\lambda_n : [0,1] \to [0,1]$, $\lambda_n(x) := x^n$, $n \in \mathbb{N} \setminus \{0\}$, the statement is a direct consequence of Proposition 2.3. □

3. Conclusions

The results presented in Section 2 can generate a lot of interesting problems. We list below three problems which are in fact direct applications of Propositions 2.1, 2.2 and 2.3, respectively.

Problem 3.1. *Let* $\alpha \in (0,1)$ *and* $J_n(\alpha) := \int_0^\alpha \ln(1 + \sin x + ... + \sin^{n-1} x)dx$ *for every* $n \geq 2$. *Calculate* $\lim_{n\to\infty} J_n(\alpha)$.

Problem 3.2. *Let* $f : [0,1] \to \mathbb{R}$ *be a differentiable function with bounded derivative,* $g : [0,1] \to \mathbb{R}$ *a Riemann integrable function which admits a primitive,* $\beta \in (0,1)$ *and let* a, b *and* c *three real constants such that*

$$\ln(1 + a + bn + cn^2) \leq n, \forall\, n \geq \mathbb{N} \setminus \{0\}.$$

Then

$$\lim_{n\to\infty} n^\beta \cdot \int_0^1 \left[f\left(\frac{\ln(1 + ax^2 + bxn + cn^2)}{n} \right) - f\,(0) \right] g(x)dx = 0.$$

Problem 3.3. *Let* $f : [0,1] \to \mathbb{R}$ *be a continuous function , differentiable at* 0, *let* $g : [0,1] \to \mathbb{R}$ *be a Riemann integrable function, continuous at* 1, *with* $g(1) = 0$, *and let* $(\alpha_n)_{n\geq 1}$ *be a sequence of positive numbers such that the sequence* $\left(\frac{\alpha_n}{n}\right)_{n\geq 1}$ *is bounded.*
Then

$$\begin{aligned} &\lim_{n\to\infty} \alpha_n \int_0^1 (f(\sin^n x) - f(0)) \cdot g(x)dx \\ = \quad & f'(0) \cdot \lim_{n\to\infty} \alpha_n \int_0^1 \sin^n x \cdot g(x)dx = 0. \end{aligned}$$

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